The Noether–Bessel-Hagen symmetry approach for dynamical systems

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Received 26 August 2020

Accepted 30 September 2020

Published 20 October 2020

The Noether–Bessel-Hagen theorem can be considered a natural extension of Noether Theorem to search for symmetries. Here, we develop the approach for dynamical systems introducing the basic foundations of the method. Specifically, we establish the Noether–Bessel-Hagen analysis of mechanical systems where external forces are present. In the second part of the paper, the approach is adopted to select symmetries for a given systems. In particular, we focus on the case of harmonic oscillator as a testbed for the theory, and on a cosmological system derived from scalar–tensor gravity with unknown

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scalar-field potential $V(\varphi)$. We show that the shape of potential is selected by the presence of symmetries. The approach results particularly useful as soon as the Lagrangian of a given system is not immediately identifiable or it is not a Lagrangian system.

Keywords: Lagrangian; Noether symmetry approach; Noether–Bessel-Hagen symmetry; invariant differential form; fibered mechanics; extended gravity cosmology.

Mathematics Subject Classification 2020: 70H33, 58E30, 58D19, 53A15

1. Introduction

Noether symmetries have proved to be one of the most prolific mathematical tools for identifying conserved quantities and reducing dynamical systems. However, sometimes, it is difficult to identify the Lagrangian or the Hamiltonian related to a given system and then apply the so-called Noether Symmetry Approach (cf. [1, 2]).

Our aim, in this paper, is to analyze Noether-type symmetries of dynamical systems with *external forces*, which leave *invariant* the corresponding equations of motion, known as the *Noether–Bessel-Hagen symmetries* [3]; for symmetries in local variational theories see monographs by Kossmann-Schwarzbach [4] and Krupka [5], and recent papers, e.g. [6–12]. As we will show, this is a straightforward generalization of the Noether Approach that can be extremely useful in several areas of physics like mechanics, field theory, cosmology and, in general, dynamical systems.

Let us recall the well-known fact that any symmetry of a Lagrangian λ is also a symmetry of the associated Euler–Lagrange form E_λ or, in other words, any Noether symmetry (assumed by the *Noether’s first theorem* [13]) of a Lagrangian $\lambda = \mathcal{L}dt$ is also a Noether–Bessel-Hagen symmetry of the mechanical system derivable from λ .

For *variational* systems, that is systems derivable as the Euler–Lagrange equations of a Lagrangian, we have both the Noether currents for invariant Lagrangians and an extension of Noether currents for invariant Euler–Lagrange forms; to this purpose, the infinitesimal first variation formula is utilized. However, mechanical systems are very often *not* variational, typical examples of those include mechanical systems with frictional forces. For non-variational mechanical systems, we may nevertheless introduce the concept of invariance and study the Noether–Bessel-Hagen symmetries, although the standard Noether conserved currents are not at disposal in this case.

A new idea of this work consists in solving the *Noether–Bessel-Hagen equation* of Killing type,

$$\partial_\xi \varepsilon = 0, \quad (1)$$

with respect to a vector field ξ , an external force ϕ and a potential function $\mathcal{U}$, which are unknown. Here, the mechanical system is described by the source form ε with components

$$\varepsilon_i = E_i(\mathcal{L}) - \phi_i,$$

given by the Euler–Lagrange expressions $E_i(\mathcal{L})$ of a mechanical Lagrange function $\mathcal{L} = \mathcal{T} - \mathcal{U}$ and force components ϕ_i . In particular, this viewpoint enable us to search for such mechanical system potentials arising from symmetry requirements.

In Sec. 2, we summarize basic definitions and results on *invariant* differential forms, defined on jet prolongations of a fibered manifold over one-dimensional base (*fibered mechanics*). As the subject of this paper concerns mechanical systems, we may reasonably restrict ourselves to *second-order* equations of motion corresponding to *first-order* Lagrangians. For general, higher-order *field theory* treatment on the subject and our main sources for the concept of invariance, we refer to works by Trautman [15, 16] and Krupka [5, 14, 17], where all proofs can be found so we omit them in this section. For geometric invariance approach see also Sardanashvily [18], and within classical Euclidean framework, Olver [19], Bluman and Kumei [20] for symmetries of differential equations, and Kossmann-Schwarzbach [4] for Noether invariant variational structures.

Section 3 contains results on symmetries of mechanical systems on with external forces. The concept of a force is introduced as a 1-form on the tangent bundle TM (cf. [21]). Our main result, formulated in Theorem 8, describes necessary and sufficient conditions for a vector field ξ , external force 1-form ϕ and potential function $\mathcal{U} : M \rightarrow \mathbb{R}$ such that Eq. (1) is satisfied identically. These conditions include the Killing equation for ξ with respect to a metric field g defining the kinetic part of the Lagrangian, and another condition on Lie derivatives of ϕ and $\mathcal{U}$; note that the proof is based on the implication “if metric g is invariant, then the Levi-Civita connection ${}^g\Gamma$ associated with g is also invariant”. Moreover, this theorem is further developed for particular forms of external force ϕ , including conservative, dissipative, or variational forces. Closely related works on the subject of symmetries and conservation laws for dissipative mechanical systems are Chien *et al.* [22, 23], using a different method based on variational multipliers (“Neutral action method”).

In Sec. 4, the theory is applied to two examples: the m -dimensional damped harmonic oscillator and a cosmological model derived from scalar–tensor theory of gravity. This latter example is particularly important for several reasons that will be discussed in Sec. 4.2. It is also worth noticing that the application of Noether Theorem to different cosmological Lagrangians is an open issue aimed at selecting, by the existence of symmetries, physically motivated models. In fact, symmetries are able to reduce dynamics and provide exact solutions of the field equations. For instance in [24–27], the Noether Symmetry Approach is applied to some modified theories of gravity in a spherically symmetric background, while in [28–31], cosmological symmetries are studied. The approach applied in these papers (whose general features can be found in [1, 2]) needs a Lagrangian description to be performed; this is the main difference with respect to the approach we are going to present in this paper, which can be directly applied to the field equations and does not require the system to be variational.

In this paper, the configuration space of a mechanical system is considered to be the second jet prolongation J^2Y of a product fibered manifold $Y = \mathbb{R} \times M$, where M is an open subset of Euclidean space $\mathbb{R}^m$. This assumption allows us to work with a global chart. Jet spaces are then canonically identified with products, namely $J^1Y \cong \mathbb{R} \times TM$, $J^2Y \cong \mathbb{R} \times T^2M$, where TM is the tangent space over M , endowed with global coordinates $(x^i, \dot{x}^i)$, and T^2M is the bundle of velocities of order 2 over M , endowed with global coordinates $(x^i, \dot{x}^i, \ddot{x}^i)$. Throughout, the Einstein summation convention is applied; $d\eta$, $i_\xi\eta$, $\partial_\xi\eta$ denote the exterior derivative, the contraction operation and the Lie derivative with respect to a vector field ξ of a differential form η . Γ_{jk}^i are the *Christoffel symbols*, associated with a metric tensor g_{ij} , i.e.

$$\Gamma_{jk}^i = \frac{1}{2}g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right). \quad (2)$$

Along the paper, $\partial_\xi g$ and $\partial_\xi \Gamma$ denote Lie derivatives of metric field g and of connection Γ with respect to a vector field ξ ,

$$(\partial_\xi \Gamma)_\eta(\zeta) = (\partial_\xi \Gamma)_{jk}^i \zeta^j \eta^k \left(\frac{\partial}{\partial x^i} \right)_x,$$

where

$$(\partial_\xi \Gamma)_{jk}^i = \frac{\partial \Gamma_{jk}^i}{\partial x^s} \zeta^s - \Gamma_{jk}^s \frac{\partial \xi^i}{\partial x^s} + \Gamma_{ks}^i \frac{\partial \xi^s}{\partial x^j} + \Gamma_{js}^i \frac{\partial \xi^s}{\partial x^k} + \frac{\partial^2 \xi^i}{\partial x^j \partial x^k}. \quad (3)$$

We use concepts and methods of global variational geometry, as described in Sec. 2, and our results can be generalized to arbitrary smooth fibered manifolds, including higher-order fibered mechanics and field theory.

2. Invariant Source Forms in Fibered Mechanics and the Noether–Bessel-Hagen Equation

Throughout, we consider a fibered manifold $\pi : Y \rightarrow X$, where $\dim X = 1$, and its *first* and *second jet prolongations* J^1Y and J^2Y , respectively. The variational geometry structures, well adapted to this work, can be found in [5, 32–34]. For an open set $W \subset Y$, denote W^1 (respectively, W^2) the preimage of W in the canonical jet projection $\pi^{1,0} : J^1Y \rightarrow Y$ (respectively, $\pi^{2,0} : J^2Y \rightarrow Y$). $\Omega^1 W$ (respectively, $\Omega^2 W$) denotes the exterior algebra of differential forms on W^1 (respectively, W^2). If (V, ψ) , $\psi = (t, x^i)$, is a fibered chart on Y , the associated chart on J^1Y (respectively, J^2Y) reads (V^1, ψ^1) , $\psi^1 = (t, x^i, \dot{x}^i)$ (respectively, (V^2, ψ^2) , $\psi^2 = (t, x^i, \dot{x}^i, \ddot{x}^i)$). By means of chart, we put $hdt = dt$, $hdx^i = \dot{x}^i dt$, $hdx^i = \ddot{x}^i dt$, and for any function $f : W^1 \rightarrow \mathbb{R}$, $hf = f \circ \pi^{2,1}$, where $\pi^{2,1} : J^2Y \rightarrow J^1Y$ is the canonical jet projection. These formulae define a *global* homomorphism of exterior algebras $h : \Omega^1 W \rightarrow \Omega^2 W$, called π -*horizontalization*. A 1-form $\rho \in \Omega_1^1 W$ is called *contact*, if $h\rho = 0$. In a fibered chart, a contact 1-form ρ is expressible as a linear combination

$\rho = A_i \omega^i$ of contact 1-forms $\omega^i = dx^i - \dot{x}^i dt$. Any differential 1-form $\rho \in \Omega_1^1 W$ has a unique decomposition $(\pi^{2,1})^* \rho = h\rho + p\rho$, where $h\rho$, respectively, $p\rho$, is the *horizontal component*, respectively, *contact component*, of ρ . Note that this decomposition has a generalization to arbitrary k -forms, see [5]. A vector field ξ on $W \subset Y$ is said to be π -projectable, if there exists a vector field ξ_0 on X such that $T\pi \circ \xi = \xi_0 \circ \pi$. In a fibered chart, a π -projectable vector field ξ has the expression $\xi = \xi^0(\partial/\partial t) + \xi^i(\partial/\partial x^i)$, where $\xi^0 = \xi^0(t)$, $\xi^i = \xi^i(t, x^j)$. For a π -projectable vector field ξ , the vector field $J^r \xi$ on $W^2 \subset J^2 Y$ is defined by the formula

$$J^2 \xi(J^2 \gamma) = \left. \frac{d}{dt} J^2 \alpha_t^\xi(J^2 \gamma) \right|_{t=0},$$

where α_t^ξ is the local one-parameter group ξ , and $J^2 \alpha_t^\xi$ is the jet prolongation of automorphism α_t^ξ .

A *Lagrangian* of order 1 for Y is defined as a π^1 -horizontal 1-form λ on $W^1 \subset J^1 Y$. In a fibered chart (V, ψ) , $\psi = (t, x^i)$, on W , λ has an expression

$$\lambda = \mathcal{L} dt, \quad (4)$$

where $\mathcal{L} : W^1 \rightarrow \mathbb{R}$ is the (local) *Lagrange function*, associated with λ . The Euler–Lagrange mapping, well known in the calculus of variations, assigns to a Lagrangian λ the associated *Euler–Lagrange form* E_λ , which is a 1-contact 2-form on $W^2 \subset J^2 Y$, locally expressed as

$$E_\lambda = E_i(\mathcal{L}) \omega^i \wedge dt, \quad (5)$$

where its coefficients $E_i(\mathcal{L})$ are the *Euler–Lagrange expressions*,

$$E_i(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{x}^i} - \frac{\partial^2 \mathcal{L}}{\partial x^j \partial \dot{x}^i} \dot{x}^j - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^j \partial \dot{x}^i} \ddot{x}^j. \quad (6)$$

The Euler–Lagrange form (5) is a particular example of a *source form* ε , which is by definition a 1-contact, $\pi^{2,0}$ -horizontal 2-form on $W^2 \subset J^2 Y$. We note that a Lagrangian is a representative of a class of 1-forms and a source form is a representative of a class of 2-forms in the variational sequence of quotient sheaves over $W \subset Y$; see, e.g. [34].

Recall now the well-known *Cartan form* Θ_λ of a first-order Lagrangian λ . Suppose $\lambda \in \Omega_{1,X}^1 W$ has a chart expression (4), then Θ_λ is locally expressed by

$$\Theta_\lambda = \mathcal{L} dt + \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \omega^i. \quad (7)$$

By means of chart transformations, one can easily observe that formula (7) defines a 1-form $\Theta_\lambda \in \Omega_1^1 W$, which has the following properties:

- (i) $h\Theta_\lambda = \lambda$; and
- (ii) $i_\xi d\Theta_\lambda$ is contact 1-form for every $\pi^{1,0}$ -vertical vector field ξ on W^1 .

We note that differential forms obeying properties (i) and (ii) are called *Lepage equivalents* (forms) of a Lagrangian, see [5, 35] and references therein. The meaning of condition (i) is that variational functionals defined by Θ_λ and λ coincide. Note also that for $\lambda \in \Omega_{1,X}^1 W$, the Lepage equivalent of λ is by conditions (i) and (ii) determined in a unique way; this is *no* longer true in higher-order field theory. Moreover, the 1-contact component of the exterior derivative of Cartan form Θ_λ coincides with the Euler–Lagrange form E_λ ,

$$p_1 d\Theta_\lambda = E_\lambda. \quad (8)$$

For a π -projectable vector field ξ on $W \subset Y$, and for any section γ of $\pi : Y \rightarrow X$ with values in W , we have the *infinitesimal first variation formula*,

$$J^1 \gamma^* \partial_{J^1 \xi} \lambda = J^2 \gamma^* i_{J^2 \xi} E_\lambda + d(J^1 \gamma^* i_{J^1 \xi} \Theta_\lambda). \quad (9)$$

A diffeomorphism $\alpha : W \rightarrow Y$ is called an *invariance transformation* of λ , respectively, ε , if

$$(J^r \alpha)^* \lambda = \lambda, \quad \text{respectively, } (J^r \alpha)^* \varepsilon = \varepsilon, \quad (10)$$

where $J^r \alpha : W^r \rightarrow J^r Y$ is the r -jet prolongation of α . Note that this definition directly applies to vector fields. A π -projectable vector field ξ on Y is called a *generator of invariance transformations* of λ , respectively, ε , if its local one-parameter group α_t^ξ consists of invariance transformations of λ , respectively, ε .

Lemma 1. *Let $\lambda \in \Omega_{1,X}^1 W$ be a Lagrangian of order 1 for Y , and let ε be a source form of order 2 for Y . A π -projectable vector field ξ on $W \subset Y$ is a generator of invariance transformations of λ , respectively ε , if and only if the Lie derivative of λ , respectively ε , with respect to $J^r \xi$ vanishes, i.e.*

$$\partial_{J^1 \xi} \lambda = 0, \quad (11)$$

respectively,

$$\partial_{J^2 \xi} \varepsilon = 0. \quad (12)$$

Generators of invariance transformations of λ , respectively, ε , form a subalgebra of the algebra of vector fields on $W \subset Y$.

Equation (11) is known as the *Noether equation* (cf. [15]); (12) is the geometric formulation of the *Noether–Bessel–Hagen equation* of the calculus of variations.

The classical (first) *Noether’s Theorem*, which describes conservation law equations for an extremal of an invariant Lagrangian, is now a straightforward consequence of formula (9).

Theorem 2. *Let $\lambda \in \Omega_{1,X}^1 W$ be a Lagrangian of order 1 for Y , and let γ be an extremal for λ . Then for every generator ξ of invariance transformations of λ ,*

$$d(J^1 \gamma^* i_{J^1 \xi} \Theta_\lambda) = 0, \quad (13)$$

where Θ_λ is the Cartan form (7) of λ .

Remark 3. The infinitesimal first variation formula implies also another consequence for invariant Lagrangians. If ξ is a generator of invariance transformations of λ , and a section γ of Y satisfies the conservation law equation (13), then the Euler–Lagrange expressions of λ are *linearly dependent* along γ .

Now, let source form ε be (locally) variational, that is ε coincides with the Euler–Lagrange form E_λ for some Lagrangian $\lambda \in \Omega_{1,X}^r W$. For any diffeomorphism $\alpha : W \rightarrow Y$, the Euler–Lagrange form E_λ obeys the formula

$$J^{2r} \alpha^* E_\lambda = E_{J^r \alpha^* \lambda}, \quad (14)$$

and for any vector field ξ on M ,

$$\partial_{J^2 \xi} E_\lambda = E_{\partial_{J^1 \xi} \lambda} \quad (15)$$

(see [5]). The next lemma is an immediate consequence of formula (14).

Lemma 4. (i) *Every invariance transformation of λ is an invariance transformation of E_λ .*
(ii) *If α is an invariance transformation of E_λ , then $\lambda - J^r \alpha^* \lambda$ is a variationally trivial Lagrangian.*

Invariant transformations of the Euler–Lagrange form E_λ extends the standard Theorem 2 of Noether.

Theorem 5. *Let $\lambda \in \Omega_{1,X}^1 W$ be a Lagrangian of order 1 for Y , let γ be an extremal for λ , and let ξ be a generator of invariance transformations of the Euler–Lagrange form E_λ . Then there exists a function f on $W \subset Y$ such that*

$$d(J^1 \gamma^* (i_{J^1 \xi} \Theta_\lambda + f)) = 0. \quad (16)$$

Remark 6. Note that Theorem 5 contains conservation law (16) in a *global* form for the order of Lagrangian equal 1 only. For second and higher-order Lagrangians, (16) gives *local* conservation laws.

3. The Noether–Bessel-Hagen Symmetries and External Forces

In this section, we study symmetries in sense of the Noether–Bessel-Hagen equation (12) for source forms, representing mechanical systems with external forces.

Consider a *Lagrangian* $\lambda = \mathcal{L} dt$ for a mechanical system, where $\mathcal{L} : TM \rightarrow \mathbb{R}$ is a first-order *Lagrange function* of the form kinetic *minus* potential energy,

$$\mathcal{L} = \mathcal{T} - \mathcal{U}. \quad (17)$$

The kinetic energy $\mathcal{T}$ is a real-valued function defined on TM by the formula

$$\mathcal{T} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \quad (18)$$

where g_{ij} is a metric tensor on M . The potential energy $\mathcal{U}$ is a real-valued function defined on the configuration space M , i.e. $\mathcal{U} = \mathcal{U}(x^i)$, and describes properties

of the mechanical system. The *Euler–Lagrange expressions* associated with $\mathcal{L}$ are functions on T^2M , defined as

$$E_i(\mathcal{L}) = -\frac{\partial \mathcal{L}}{\partial x^i} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^i}. \quad (19)$$

Substituting the form (17) of $\mathcal{L}$ into (19), we get

$$\begin{aligned} E_i(\mathcal{L}) &= E_i(\mathcal{T}) + \frac{\partial \mathcal{U}}{\partial x^i} = -\frac{\partial \mathcal{T}}{\partial x^i} + \frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \dot{x}^i} + \frac{\partial \mathcal{U}}{\partial x^i} \\ &= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \dot{x}^j \dot{x}^k + g_{ij} \ddot{x}^j + \frac{\partial \mathcal{U}}{\partial x^i} \\ &= g_{ij} \left(\ddot{x}^j + \Gamma_{ks}^j \dot{x}^k \dot{x}^s \right) + \frac{\partial \mathcal{U}}{\partial x^i}, \end{aligned}$$

where Γ_{ks}^j are the *Christoffel symbols* (2) associated to g_{ij} . Note that the Lagrange function $\mathcal{L}$ (17) is regular since g_{ij} is a non-singular matrix of the form

$$g_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}.$$

Equations of motion of the mechanical system are the *Euler–Lagrange equations associated with $\mathcal{L}$* ,

$$E_i(\mathcal{L}) = 0, \quad i = 1, \dots, m. \quad (20)$$

The energy of the system equals

$$E = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \dot{x}^i - \mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \mathcal{U},$$

hence the conservation law of energy along extremals reads

$$\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \mathcal{U} = 0. \quad (21)$$

By an *external force* for the mechanical system we call any $\pi^{1,0}$ -horizontal 1-form ϕ on TM , locally expressed as

$$\phi = \phi_i dx^i, \quad (22)$$

with components $\phi_i = \phi_i(x^j, \dot{x}^j)$ smooth real-valued functions on TM .

Consider equations of motion (20) under the influence of external force $\phi = (\phi_i)$,

$$E_i(\mathcal{L}) = \phi_i, \quad i = 1, \dots, m. \quad (23)$$

System (23) associates a dynamical form ε on $\mathbb{R} \times T^2M$,

$$\varepsilon = \varepsilon_i \omega^i \wedge dt, \quad (24)$$

where

$$\varepsilon_i = E_i(\mathcal{L}) - \phi_i, \quad (25)$$

and $\omega^i = dx^i - \dot{x}^i dt$ are contact 1-forms on $\mathbb{R} \times TM$. We call ε (24) the *source form associated with Lagrangian $\lambda = \mathcal{L} dt$* (17) and force ϕ (22).

Let ξ be a vector field on M , locally expressed by

$$\xi = \xi^i \frac{\partial}{\partial x^i}. \quad (26)$$

Clearly, ξ is π -projectable, and let $J^2\xi$ be its second jet prolongation on T^2M ,

$$J^2\xi = \xi^i \frac{\partial}{\partial x^i} + \dot{\xi}^i \frac{\partial}{\partial \dot{x}^i} + \ddot{\xi}^i \frac{\partial}{\partial \ddot{x}^i}, \quad (27)$$

where

$$\dot{\xi}^i = \frac{d\xi^i}{dt} = \frac{\partial \xi^i}{\partial x^j} \dot{x}^j, \quad \ddot{\xi}^i = \frac{d^2 \xi^i}{dt^2} = \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} \dot{x}^j \dot{x}^k + \frac{\partial \xi^i}{\partial x^j} \ddot{x}^j. \quad (28)$$

The Noether symmetries of λ (17) are given by the following lemma.

Lemma 7. *The following two conditions are equivalent:*

- (a) *A vector field ξ on M is a generator of invariance transformations of λ (17).*
- (b) *ξ and $\mathcal{W}$ satisfy the Killing equations*

$$\partial_\xi g = 0, \quad \partial_\xi \mathcal{W} = 0. \quad (29)$$

Proof. Equivalence of (a) and (b) is due to invariance Lemma 1. Indeed, by a straightforward calculation we get

$$\begin{aligned} \partial_{J^1\xi} \lambda &= (\partial_{J^1\xi} \mathcal{L}) dt = \left(\frac{\partial \mathcal{L}}{\partial x^i} \xi^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \dot{\xi}^i \right) dt \\ &= \left(\frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} \xi^i + g_{ij} \frac{\partial \xi^i}{\partial x^k} + g_{ik} \frac{\partial \xi^i}{\partial x^j} \right) \dot{x}^j \dot{x}^k - \frac{\partial \mathcal{W}}{\partial x^i} \xi^i \right) dt \\ &= \left(\frac{1}{2} (\partial_\xi g) \dot{x}^j \dot{x}^k - \partial_\xi \mathcal{W} \right) dt, \end{aligned}$$

as required. $\square$

Now, we formulate our main theorem, describing symmetries of source form ε (24) associated with λ and ϕ . More precisely, our aim is to find conditions on a vector field ξ on M , a potential energy function $\mathcal{W} = \mathcal{W}(x^i)$, and force $\phi = (\phi_i)$ such that the *Noether–Bessel-Hagen equation*,

$$\partial_{J^2\xi} \varepsilon = 0, \quad (30)$$

holds (see Lemma 1, (12)).

Theorem 8. *Let ε be a source form on $\mathbb{R} \times T^2M$, associated with $\lambda = \mathcal{L} dt$ (17) and $\phi = (\phi_i)$ (22). The following two conditions are equivalent:*

- (a) *A vector field ξ on M is a generator of invariance transformations of ε , i.e. the Noether–Bessel-Hagen equation (30) is satisfied identically.*

(b) ξ , $\mathcal{W}$, and (ϕ_i) satisfy the following the conditions:

$$\partial_\xi g = 0, \quad (31)$$

and, for every i ,

$$\frac{\partial}{\partial x^i}(\partial_\xi \mathcal{W}) + (\partial_{J^1 \xi} \phi)_i = 0, \quad (32)$$

where

$$(\partial_{J^1 \xi} \phi)_i = \frac{\partial \xi^j}{\partial x^i} \phi_j + \frac{\partial \phi_i}{\partial x^j} \xi^j + \frac{\partial \phi_i}{\partial \dot{x}^j} \frac{\partial \xi^j}{\partial x^k} \dot{x}^k \quad (33)$$

represent the Lie derivative $\partial_{J^1 \xi} \phi$ of external force 1-form $\phi = (\phi_i)$ with respect to $J^1 \xi$, and $\partial_\xi \mathcal{W}$ is the Lie derivative of function $\mathcal{W}$ with respect to ξ .

Proof. Equivalence of the conditions (a) and (b) follows from chart analysis of Eq. (30). Indeed, computing the Lie derivative of source form ε (24) with respect to $J^2 \xi$ (27), we obtain

$$\begin{aligned} \partial_{J^2 \xi} \varepsilon &= \partial_{J^2 \xi} (\varepsilon_i \omega^i) \wedge dt \\ &= \left(\frac{\partial \varepsilon_i}{\partial x^j} \xi^j + \frac{\partial \varepsilon_i}{\partial \dot{x}^j} \dot{\xi}^j + \frac{\partial \varepsilon_i}{\partial \ddot{x}^j} \ddot{\xi}^j + \frac{\partial \xi^j}{\partial x^i} \varepsilon_j \right) dx^i \wedge dt \\ &= \left(\frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial g_{pq}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^q} - \frac{\partial g_{iq}}{\partial x^p} \right) \dot{x}^p \dot{x}^q - \frac{\partial g_{ip}}{\partial x^j} \dot{x}^p - \frac{\partial^2 \mathcal{W}}{\partial x^i \partial x^j} - \frac{\partial \phi_i}{\partial x^j} \right) \xi^j \\ &\quad + \left(\left(\frac{\partial g_{pj}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p} \right) \dot{x}^p - \frac{\partial \phi_i}{\partial \dot{x}^j} \right) \frac{\partial \xi^j}{\partial x^k} \dot{x}^k \\ &\quad - g_{ij} \left(\frac{\partial^2 \xi^j}{\partial x^k \partial x^l} \dot{x}^k \dot{x}^l + \frac{\partial \xi^j}{\partial x^k} \ddot{x}^k \right) \\ &\quad + \frac{\partial \xi^j}{\partial x^i} \left(\frac{1}{2} \left(\frac{\partial g_{pq}}{\partial x^j} - \frac{\partial g_{jp}}{\partial x^q} - \frac{\partial g_{jq}}{\partial x^p} \right) \dot{x}^p \dot{x}^q - g_{jp} \ddot{x}^p - \frac{\partial \mathcal{W}}{\partial x^j} - \phi_j \right) dx^i \wedge dt \\ &= \left(\frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial g_{pq}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^q} - \frac{\partial g_{iq}}{\partial x^p} \right) \xi^j \dot{x}^p \dot{x}^q \right. \\ &\quad + \frac{1}{2} \left(\left(\frac{\partial g_{qj}}{\partial x^i} - \frac{\partial g_{iq}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^q} \right) \frac{\partial \xi^j}{\partial x^p} + \left(\frac{\partial g_{pj}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p} \right) \frac{\partial \xi^j}{\partial x^q} \right) \dot{x}^p \dot{x}^q \\ &\quad + \frac{1}{2} \left(\frac{\partial g_{pq}}{\partial x^j} - \frac{\partial g_{jp}}{\partial x^q} - \frac{\partial g_{jq}}{\partial x^p} \right) \frac{\partial \xi^j}{\partial x^i} \dot{x}^p \dot{x}^q - g_{ij} \frac{\partial^2 \xi^j}{\partial x^p \partial x^q} \dot{x}^p \dot{x}^q \\ &\quad - \frac{\partial \mathcal{W}}{\partial x^j} \frac{\partial \xi^j}{\partial x^i} - \frac{\partial^2 \mathcal{W}}{\partial x^i \partial x^j} \xi^j - \phi_j \frac{\partial \xi^j}{\partial x^i} - \frac{\partial \phi_i}{\partial x^j} \xi^j - \frac{\partial \phi_i}{\partial \dot{x}^j} \frac{\partial \xi^j}{\partial x^k} \dot{x}^k \\ &\quad \left. - \left(\frac{\partial g_{ik}}{\partial x^j} \xi^j + g_{ij} \frac{\partial \xi^j}{\partial x^k} + g_{jk} \frac{\partial \xi^j}{\partial x^i} \right) \ddot{x}^k \right) dx^i \wedge dt. \end{aligned}$$

Hence, $\partial_{J^2\xi}\varepsilon$ vanishes identically if and only if the following two conditions are satisfied:

$$\frac{\partial g_{ik}}{\partial x^j}\xi^j + g_{ij}\frac{\partial \xi^j}{\partial x^k} + g_{jk}\frac{\partial \xi^j}{\partial x^i} = 0, \quad (34)$$

which is nothing but (31), and

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial g_{pq}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^q} - \frac{\partial g_{iq}}{\partial x^p} \right) \xi^j \dot{x}^p \dot{x}^q \\ & + \frac{1}{2} \left(\left(\frac{\partial g_{qj}}{\partial x^i} - \frac{\partial g_{iq}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^q} \right) \frac{\partial \xi^j}{\partial x^p} + \left(\frac{\partial g_{pj}}{\partial x^i} - \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p} \right) \frac{\partial \xi^j}{\partial x^q} \right) \dot{x}^p \dot{x}^q \\ & + \frac{1}{2} g^{rj} \left(\frac{\partial g_{jp}}{\partial x^q} + \frac{\partial g_{jq}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^j} \right) \left(\frac{\partial g_{ir}}{\partial x^s} \xi^s + g_{is} \frac{\partial \xi^s}{\partial x^r} \right) \dot{x}^p \dot{x}^q - g_{ij} \frac{\partial^2 \xi^j}{\partial x^p \partial x^q} \dot{x}^p \dot{x}^q \\ & - \frac{\partial \mathcal{U}}{\partial x^j} \frac{\partial \xi^j}{\partial x^i} - \frac{\partial^2 \mathcal{U}}{\partial x^i \partial x^j} \xi^j - \phi_j \frac{\partial \xi^j}{\partial x^i} - \frac{\partial \phi_i}{\partial x^j} \xi^j - \frac{\partial \phi_i}{\partial \dot{x}^j} \frac{\partial \xi^j}{\partial x^k} \dot{x}^k = 0. \end{aligned}$$

Applying the following standard identities:

$$\frac{\partial g_{ij}}{\partial x^k} = g_{is} \Gamma_{jk}^s + g_{js} \Gamma_{ik}^s, \quad g_{ik} \Gamma_{pq}^k = \frac{1}{2} \left(\frac{\partial g_{ip}}{\partial x^q} + \frac{\partial g_{iq}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^i} \right),$$

the latter condition can be also rewritten as

$$g_{ik} (\partial_\xi \Gamma)_{pq}^k \dot{x}^p \dot{x}^q + \frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{U}}{\partial x^j} \xi^j \right) + \phi_j \frac{\partial \xi^j}{\partial x^i} + \frac{\partial \phi_i}{\partial x^j} \xi^j + \frac{\partial \phi_i}{\partial \dot{x}^j} \frac{\partial \xi^j}{\partial x^k} \dot{x}^k = 0, \quad (35)$$

where

$$(\partial_\xi \Gamma)_{pq}^k = \frac{\partial \Gamma_{pq}^k}{\partial x^j} \xi^j - \Gamma_{pq}^j \frac{\partial \xi^k}{\partial x^j} + \Gamma_{qj}^k \frac{\partial \xi^j}{\partial x^p} + \Gamma_{pj}^k \frac{\partial \xi^j}{\partial x^q} + \frac{\partial^2 \xi^k}{\partial x^p \partial x^q} \quad (36)$$

represent the Lie derivative $\partial_\xi \Gamma$ (3) of the Levi-Civita connection $\Gamma = (\Gamma_{pq}^k)$ with respect to ξ . However, since ${}^g\Gamma$ is invariant provided g is invariant, it follows from (34) that also (36) vanishes. Thus, (35) already implies (32). This proves equivalence of the conditions (a) and (b). $\square$

Reducing the form of external force $\phi = (\phi_i)$, we obtain important simplifications of Theorem 8. Consider the following cases:

(I) ϕ is identically zero, i.e. (23) coincides with the Euler–Lagrange equations, associated with $\lambda = \mathcal{L}dt$.

(II) ϕ is *conservative*, i.e. its components ϕ_i are of the form

$$\phi_i = -\frac{\partial \mathcal{U}}{\partial x^i}. \quad (37)$$

(III) ϕ is defined on M , i.e. its components ϕ_i are functions of x^j variables only.

(IV) ϕ is *dissipative*, i.e. its components ϕ_i are of the form

$$\phi_i = -\phi_{ij}\dot{x}^j, \quad (38)$$

where $\phi_{ij} : M \rightarrow \mathbb{R}$ is a symmetric matrix, and $\phi_i = -\partial F/\partial \dot{x}^i$ for a *dissipative function*,

$$F = \frac{1}{2}\phi_{ij}\dot{x}^i\dot{x}^j, \quad (39)$$

which is a positive definite quadratic form in the dot variables (cf. [36]).

(V) ϕ is *variational*, i.e. ϕ_i coincide with the Euler–Lagrange expressions of a Lagrange function, hence

$$\phi_i = \frac{\partial h}{\partial x^i} + \left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} \right) \dot{x}^j, \quad (40)$$

for some functions $h = h(x^j)$, $\eta_i = \eta_i(x^j)$, on M (cf. [21]), which are free parameters. An alternative approach using variational multipliers has been applied in [23, 22].

Remark 9. For a variational external force ϕ (Case (V)), the source form ε (24), associated to $\lambda = \mathcal{L}dt$ (17) and ϕ is again variational and it coincides with the Euler–Lagrange form $E_{\tilde{\lambda}}$ of the Lagrangian $\tilde{\lambda} = \tilde{\mathcal{L}}dt$, where

$$\tilde{\mathcal{L}} = \mathcal{L} - h + \eta_j \dot{x}^j. \quad (41)$$

Note also that the external force $\tilde{\phi} = \tilde{\phi}_i dx^i$, where

$$\tilde{\phi}_i = \phi_i - \frac{\partial h}{\partial x^i} = \left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} \right) \dot{x}^j, \quad (42)$$

is dissipative, see Case (IV) for

$$\phi_{ij} = \frac{\partial \eta_j}{\partial x^i} - \frac{\partial \eta_i}{\partial x^j}.$$

Corollary 10. Let ε be a source form on $\mathbb{R} \times T^2M$, associated with $\lambda = \mathcal{L}dt$ (17) and $\phi = (\phi_i)$, given by Cases (I)–(V). A vector field ξ on M is a generator of invariance transformations of ε if and only if ξ , $\mathcal{W}$, and $\phi = (\phi_i)$ satisfy

(Case (I))

$$\partial_\xi g = 0, \quad \partial_\xi \mathcal{W} = \text{const.} \quad (43)$$

(Case (II))

$$\partial_\xi g = 0. \quad (44)$$

(Case (III))

$$\partial_\xi g = 0, \quad (\partial_\xi \phi)_i + \frac{\partial}{\partial x^i}(\partial_\xi \mathcal{W}) = 0. \quad (45)$$

(Case (IV))

$$\partial_{\xi}g = 0, \quad \partial_{J^1\xi}\phi = 0, \quad \partial_{\xi}\mathcal{U} = \text{const.} \quad (46)$$

(Case (V))

$$\partial_{\xi}g = 0, \quad \partial_{J^1\xi}\tilde{\phi} = 0, \quad \partial_{\xi}(\mathcal{U} + h) = \text{const.}, \quad (47)$$

where $\tilde{\phi}$ is 1-form given by (42).

Proof. Conditions (43)–(47) follow from Theorem 8, where we substitute the form of external force $\phi = (\phi_i)$ with respect to Cases (I)–(V). $\square$

The Noether Theorem (Theorem 2) and its extension (Theorem 5) have the following implications for the mechanical Lagrangian (17) and for its completion by means of a variational external force.

Corollary 11. *Let E_{λ} be the Euler–Lagrange form, associated to Lagrangian $\lambda = \mathcal{L}dt$ (17). Let ξ be a vector field on M , and γ be an extremal for λ . Then*

(i) *For every generator ξ (26) of invariance transformations of λ ,*

$$dJ^1\gamma^*(g_{ij}\xi^i\dot{x}^j) = 0, \quad (48)$$

i.e. γ is a geodesic with respect to the Levi-Civita connection ${}^g\Gamma$.

(ii) *For every generator ξ (26) of invariance transformations of E_{λ} ,*

$$dJ^1\gamma^*(g_{ij}\xi^i\dot{x}^j + \mathcal{U}_0t) = 0, \quad (49)$$

where $\mathcal{U}_0 \in \mathbb{R}$ is given by

$$\mathcal{U}_0 = \frac{\partial \mathcal{U}}{\partial x^j}\xi^j.$$

Proof. (1) Suppose ξ is a generator of invariance transformations of λ . Substituting the form of Lagrangian λ (17) and the expression of the Cartan form,

$$\Theta_{\lambda} = \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i \right) dt + \frac{\partial \mathcal{L}}{\partial \dot{x}^i}dx^i,$$

into Theorem 2, we easily get the Noether current given by (48). Moreover, using condition $\partial_{\xi}g = 0$ (29) from Lemma (7), we obtain

$$\begin{aligned} dJ^1\gamma^*(g_{ij}\xi^i\dot{x}^j) &= J^1\gamma^*d(g_{ij}\xi^i\dot{x}^j) \\ &= J^2\gamma^*\left(\left(\frac{\partial g_{ij}}{\partial x^k}\xi^i + g_{ij}\frac{\partial \xi^i}{\partial x^k}\right)\dot{x}^j\dot{x}^k + g_{ik}\xi^i\ddot{x}^k\right)dt \\ &= J^2\gamma^*(g_{ik}\xi^i(\ddot{x}^k + \Gamma_{js}^k\dot{x}^j\dot{x}^s))dt. \end{aligned}$$

Hence, (48) holds if and only if γ is a geodesic with respect to (Γ_{jk}^i) .

(2) Suppose ξ is a generator of invariance transformations of E_λ . Since γ is an extremal for λ , the first variation formula (9) reduces to

$$J^1\gamma^*\partial_{J^1\xi}\lambda = d(J^1\gamma^*i_{J^1\xi}\Theta_\lambda). \quad (50)$$

From Corollary 10 (Case (I)), we have $\partial_\xi g = 0$, and

$$\frac{\partial\mathcal{U}}{\partial x^j}\xi^j = \mathcal{U}_0 \in \mathbb{R},$$

hence the Lie derivative $\partial_{J^1\xi}\lambda$ of Lagrangian (17) reads

$$\begin{aligned} \partial_{J^1\xi}\lambda &= \left(\left(\frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k - \frac{\partial\mathcal{U}}{\partial x^i} \right) \xi^i + g_{ij} \dot{x}^j \dot{\xi}^i \right) dt \\ &= \left(\frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} \xi^i + g_{ij} \frac{\partial \xi^i}{\partial x^k} + g_{ik} \frac{\partial \xi^i}{\partial x^j} \right) \dot{x}^j \dot{x}^k - \frac{\partial\mathcal{U}}{\partial x^i} \xi^i \right) dt \\ &= -\mathcal{U}_0 dt. \end{aligned}$$

Since $\partial_{J^2\xi}E_\lambda$ vanishes, we see at once that $\partial_{J^1\xi}\lambda$ is a variationally trivial Lagrangian by means of formula (15). Applying this expression into (50), we get

$$J^2\gamma^* \left(\mathcal{U}_0 + g_{ik} \xi^i (\ddot{x}^k + \Gamma_{js}^k \dot{x}^j \dot{x}^s) \right) dt = 0,$$

or, equivalently, (49). This result completes Theorem 5 for the mechanical Lagrangian $\lambda = (\mathcal{T} - \mathcal{U})dt$ (17) whose Lie derivative $\partial_{J^1\xi}\lambda$ is of order zero. $\square$

Corollary 12. *Let ε (24) be the source form, associated to Lagrangian $\lambda = \mathcal{L}dt$ (17) and variational force $\phi = (\phi_i)$ (40). Let ξ be a vector field on M , and γ be an extremal for $\tilde{\lambda}$ (41). Then*

(i) *For every generator ξ (26) of invariance transformations of $\tilde{\lambda}$,*

$$dJ^1\gamma^* \left((g_{ij}\dot{x}^j + \eta_i) \xi^i \right) = 0. \quad (51)$$

(ii) *For every generator ξ (26) of invariance transformations of $E_{\tilde{\lambda}}$,*

$$dJ^1\gamma^* ((g_{ij}\dot{x}^j + \eta_i)\xi^i + \mathcal{U}_0 t - f) = 0, \quad (52)$$

where $\mathcal{U}_0 = \partial_\xi \mathcal{U} \in \mathbb{R}$ and f is a solution of the first-order partial differential equation

$$\frac{\partial \eta_j}{\partial x^i} \xi^i + \eta_i \frac{\partial \xi^i}{\partial x^j} = \frac{\partial f}{\partial x^j}, \quad (53)$$

which is integrable, and on a star-shaped domain, a solution of (53) reads

$$f = x^j \int_0^1 \left(\frac{\partial \eta_j}{\partial x^i} \xi^i + \eta_i \frac{\partial \xi^i}{\partial x^j} \right)_{(sx^k)} ds.$$

Proof. (1) Analogously as in Corollary 11, formula (51) follows from the Noether Theorem 2.

(2) Suppose ξ is a generator of invariance transformations of $E_{\tilde{\lambda}}$ and γ is an extremal for $\tilde{\lambda}$. Applying Corollary 10 (Case (V)), namely the identities $\partial_{\xi}g = 0$, and $\partial_{\xi}(\mathcal{U} + h) = \mathcal{U}_0 \in \mathbb{R}$, we get

$$\partial_{J^1\xi}\tilde{\lambda} = \left(\left(\frac{\partial\eta_j}{\partial x^i}\xi^i + \eta_i \frac{\partial\xi^i}{\partial x^j} \right) \dot{x}^j - \mathcal{U}_0 \right) dt.$$

However, it easy to see that the identity $\partial\tilde{\phi} = 0$ (47) is an integrability condition of the equation

$$\left(\frac{\partial\eta_j}{\partial x^i}\xi^i + \eta_i \frac{\partial\xi^i}{\partial x^j} \right) \dot{x}^j = \frac{df}{dt}$$

for unknown function $f = f(x^k)$. Computing the contraction of the Cartan form $i_{J^1\xi}\Theta_{\tilde{\lambda}}$, we get from the first variation formula (9) the Noether conservation law (52). $\square$

4. Examples

4.1. The damped harmonic oscillator

Consider the equations of motion of the *damped oscillator* in $\mathbb{R} \times M$, where M is an open subset of the Euclidean space $\mathbb{R}^m$,

$$m_{ij}\ddot{x}^j + k_{ij}x^j = -\phi_{ij}\dot{x}^j, \quad (54)$$

$i = 1, \dots, m$, where both the mass coefficients m_{ij} and the potential energy coefficients k_{ij} are given by symmetric matrices over the field of real numbers $\mathbb{R}$. Moreover, we assume that m_{ij} is a non-singular; if this is not the case, we simply relax the *regularity condition* on the metric tensor g , cf. Sec. 3 (18). System (54) belongs to the class of mechanical systems with dissipative external forces, cf. (38). The left-hand sides of (54) are the Euler–Lagrange expressions of the *free oscillator* Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{1}{2}(m_{ij}\dot{x}^i\dot{x}^j - k_{ij}x^ix^j), \quad (55)$$

see (17), and the *energy* of the system equals

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\dot{x}^i - \mathcal{L} = \mathcal{T} + \mathcal{U} = \frac{1}{2}(m_{ij}\dot{x}^i\dot{x}^j + k_{ij}x^ix^j). \quad (56)$$

The external force $\phi = (\phi_i)$, see (22), (38), is given by

$$\phi_i = -\phi_{ij}\dot{x}^j, \quad (57)$$

also called the generalized frictional forces, see [36]. Note that system (54) is *not* variational unless $\phi_{ij} = 0$, which contradicts the positive definiteness of the dissipative function (39). Source form ε , associated with Lagrange function (55) and

dissipative force ϕ , is expressed by $\varepsilon = \varepsilon_i \omega^i \wedge dt$ (24), where

$$\varepsilon_i = m_{ij} \ddot{x}^j + k_{ij} x^j + \phi_{ij} \dot{x}^j. \quad (58)$$

Let ξ be a vector field on M , locally expressed by (26), and let $J^2 \xi$ (27), (28), be its second jet prolongation. Applying Corollary 10 of Theorem 8, we obtain conditions on symmetry ξ on M , potential coefficients k_{ij} , and dissipative force coefficients ϕ_{ij} such that the Noether–Bessel–Hagen equation (30) for source form ε holds.

Theorem 13. *Let ε (58) be a source form on $\mathbb{R} \times T^2 M$, associated to the free oscillator Lagrangian $\lambda = \mathcal{L} dt$ (55) and a dissipative force $\phi = (\phi_1, \phi_2)$ (57). Let ξ be a vector field on M , expressed by (26).*

(a) *ξ is a generator of invariance transformations of λ if and only if*

$$m_{ip} \frac{\partial \xi^i}{\partial x^q} + m_{iq} \frac{\partial \xi^i}{\partial x^p} = 0, \quad (59)$$

and

$$k_{ij} x^j \xi^i = 0. \quad (60)$$

The Noether conserved current (Corollary 11, (48)) along extremals for the free harmonic oscillator (i.e. $\phi = 0$) reads

$$L = m_{ij} \xi^i \dot{x}^j. \quad (61)$$

(b) *ξ is a generator of invariance transformations of ε if and only if ξ^i and ϕ_{ij} satisfy (59) and*

$$k_{ij} \xi^j + k_{js} x^s \frac{\partial \xi^j}{\partial x^i} = 0, \quad (62)$$

$$\phi_{is} \frac{\partial \xi^s}{\partial x^j} + \phi_{js} \frac{\partial \xi^s}{\partial x^i} + \frac{\partial \phi_{ij}}{\partial x^s} \xi^s = 0, \quad (63)$$

For $\phi = 0$, along extremals for the free harmonic oscillator the Noether–Bessel–Hagen conserved current (Corollary 11, (49)) reads

$$L = m_{ij} \xi^i \dot{x}^j + k_{ij} x^i \xi^j t.$$

Proof. Assertions (a) and (b) are reformulations of Lemma 7 and Theorem 8, respectively. Indeed, conditions (59), correspond to $\partial_{\xi} g = 0$, (60), (62) to $\partial_{\xi} \mathcal{U} = 0$, and (63) corresponds to $\partial_{J^1 \xi} \phi = 0$, cf. Corollary 10, (46). $\square$

Remark 14. In the case of the Euclidean metric $m_{ij} = k_{ij} = \delta_{ij}$, conditions (59), (60) possess a unique solution of the form $\xi^i = P_j^i x^j$, where P_j^i are some real numbers such that $P_j^i = -P_i^j$. The Noether current (61) is $L = \delta_{ij} \xi^i \dot{x}^j$. If $m = 3$,

for the three-dimensional free oscillator L is expressed as

$$\begin{aligned} L &= \sum_{j=1}^3 \xi^j \dot{x}^j \\ &= -P_2^1 r^2 \dot{\varphi} \sin^2 \vartheta + P_3^1 r^2 (\dot{\vartheta} \cos \varphi - \dot{\varphi} \sin \vartheta \cos \vartheta \sin \varphi) \\ &\quad + P_3^2 r^2 (\dot{\vartheta} \sin \varphi + \dot{\varphi} \sin \vartheta \cos \vartheta \cos \varphi), \end{aligned}$$

which is a linear combination of components of the *angular momentum* in the spherical coordinates (r, φ, ϑ) on M .

Conditions (59), (62), (63), with respect to the Euclidean metric, possess a unique solution of the form $\xi^i = P_j^i x^j$, where $P_j^i \in \mathbb{R}$ satisfy $P_j^i = -P_i^j$, and

$$\phi_{is} P_j^s + \phi_{js} P_i^s + \frac{\partial \phi_{ij}}{\partial x^s} P_l^s x^l = 0.$$

For $\phi = 0$, the Noether–Bessel-Hagen conserved current (49) reads

$$L = \sum_j \xi^j x^j + \sum_j \xi^j x^j t$$

and it coincides with the angular momentum L since $\xi^j x^j$ vanishes.

4.2. *Scalar–tensor cosmological models in two-dimensional configuration space*

Extended and modified theories of gravity [56, 60] acquired a lot of interest in the recent years due to some inconsistencies provided by Einstein General Relativity which presents some shortcomings at ultraviolet and infrared scales. As an example, a grand unification theory in which gravity and the other fundamental interactions are included is missing so far, if one considers General Relativity as the final theory describing the gravitational force. More fundamentally, finding a link between General Relativity and Quantum Mechanics is one of the main goal of recent physics, and, despite of many attempts, Einstein’s theory seems to be not the best candidate to this purpose. According to this point of view, many other approaches have been developed as *String Theory* [37–41], *Kaluza–Klein Theory* [42, 43], *Loop Quantum Gravity* [44–46], *Horava–Lifshitz Gravity* [47–50], *Non-Local Gravity* [51–53], etc. While General Relativity turns out to be non-renormalizable from the two-loop level, the above mentioned theories are both renormalizable and unitary.

However, ultra-violet scale issues do not represent the only problems of General Relativity: at the low-scale regime there are several incongruities between theory and observations emerging at astrophysical and cosmological scales. Some of them are the anomalous accelerated expansion of the universe, the flat rotation curves of galaxies, the dynamics of cluster of galaxies, etc. These shortcomings can be overcome by assuming huge amount of dark energy and dark matter or relaxing the strict hypothesis that General Relativity is the theory describing

gravitational interaction at any scale. Proposals are modifying the Hilbert–Einstein action [54, 55], including the torsion [56–58], and studying the related dynamics [59–61, 42].

Another issue of General Relativity is related the early cosmology; the accepted paradigm is that of *inflation* which involves a scalar field (or more than one scalar field), the *inflation* φ , to generate the primordial accelerated phase capable of addressing the shortcomings of Cosmological Standard Model [62–64]. In these models, gravity is assumed minimally or non-minimally coupled to φ . Theories in which the action contains both the Ricci scalar, the kinetic term $\dot{\varphi}^2$, the potential $V(\varphi)$ and a coupling $F(\varphi)$ are called *scalar–tensor* theories of gravity [65, 66]. A fundamental question is to select theories with reliable $F(\varphi)$ and $V(\varphi)$ capable of giving realistic cosmological models to be compared with observational data. The Noether Symmetry Approach proved extremely useful in selecting physically motivated model. See [1] for a review.

In this section, we apply the preceding Noether–Bessel-Hagen approach to the dynamical equations arising in a particular scalar–tensor cosmological model. As said above, finding the form of the potential from symmetries is important to derive conserved quantities, to reduce the dynamics and finally to find exact solutions that are always physically motivated [1]. Some examples can be found in [67–71].

Let us adopt the above method for scalar–tensor cosmological models where external forces appear into dynamic with the aim to select by symmetries the scalar-field potential and possible interacting terms. See for example [72].

Let M be an open subset of the Euclidean plane $\mathbb{R}^2$. The only coordinates are the cosmological scale factor a and the scalar field φ , which form a global chart on $M \subset \mathbb{R}^2$. Consider the Klein–Gordon Lagrange function on TM , given by

$$\mathcal{L}(a, \varphi, \dot{a}, \dot{\varphi}) = 3a\dot{a}^2 - a^3 \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi) \right),$$

where V is a smooth function depending on φ only as standard in cosmology [62–71]. This Lagrange function is of the form (17),

$$\mathcal{L} = \frac{1}{2}g_{11}\dot{a}^2 + g_{12}\dot{a}\dot{\varphi} + \frac{1}{2}g_{22}\dot{\varphi}^2 - \mathcal{U}, \quad (64)$$

where

$$g_{11} = 6a, \quad g_{12} = 0, \quad g_{22} = -a^3, \quad \mathcal{U}(a, \varphi) = -a^3V(\varphi).$$

The associated Euler–Lagrange expressions are

$$\begin{aligned} -\frac{\partial \mathcal{L}}{\partial a} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} &= -3\dot{a}^2 + 3a^2 \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi) \right) + \frac{d}{dt} (6a\dot{a}) \\ &= 6a\ddot{a} + 3\dot{a}^2 + 3a^2 \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi) \right), \end{aligned} \quad (65)$$

$$\begin{aligned}
 -\frac{\partial \mathcal{L}}{\partial \varphi} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &= -a^3 \frac{dV}{d\varphi} + \frac{d}{dt} (-a^3 \dot{\varphi}) \\
 &= -a^3 \frac{dV}{d\varphi} - 3a^2 \dot{a} \dot{\varphi} - a^3 \ddot{\varphi},
 \end{aligned} \tag{66}$$

and the *energy condition* is

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} &= 6a\dot{a}^2 - a^3 \dot{\varphi}^2 - 3a\dot{a}^2 + a^3 \left(\frac{1}{2} \dot{\varphi}^2 - V(\varphi) \right) \\
 &= 3a\dot{a}^2 - a^3 \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right).
 \end{aligned} \tag{67}$$

Hence, the Euler–Lagrange equations read

$$6a\ddot{a} + 3\dot{a}^2 + \frac{3}{2}a^2\dot{\varphi}^2 - 3a^2V(\varphi) = 0, \tag{68}$$

$$-a^3\ddot{\varphi} - 3a^2\dot{a}\dot{\varphi} - a^3\frac{dV}{d\varphi} = 0. \tag{69}$$

Note that this system is equivalent to

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{2}\dot{\varphi}^2 - V(\varphi) = 0, \tag{70}$$

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{71}$$

which is the system of equations in [68], with $F(\varphi) = 1/2$. However, (70), (71) is *not* variational in the sense discussed above.

Consider the equations of motion defined by (65), (66), under influence of an external force (f_a, f_φ) ,

$$6a\ddot{a} + 3\dot{a}^2 + \frac{3}{2}a^2\dot{\varphi}^2 - 3a^2V(\varphi) = \phi_a, \tag{72}$$

$$-a^3\ddot{\varphi} - 3a^2\dot{a}\dot{\varphi} - a^3\frac{dV}{d\varphi} = \phi_\varphi, \tag{73}$$

where $\phi_a = \phi_a(a, \varphi, \dot{a}, \dot{\varphi})$, $\phi_\varphi = \phi_\varphi(a, \varphi, \dot{a}, \dot{\varphi})$ are smooth real-valued functions on TM . The source form ε , associated with Lagrangian $\lambda = \mathcal{L}dt$ (64) and force $\phi = \phi_a da + \phi_\varphi d\varphi$ (cf. (22)), is

$$\varepsilon = (\varepsilon_a \omega^a + \varepsilon_\varphi \omega^\varphi) \wedge dt, \tag{74}$$

where its coefficients $\varepsilon_a, \varepsilon_\varphi$ are real-valued functions on T^2M , given by

$$\varepsilon_a = 6a\ddot{a} + 3\dot{a}^2 + \frac{3}{2}a^2\dot{\varphi}^2 - 3a^2V(\varphi) - \phi_a,$$

$$\varepsilon_\varphi = -a^3\ddot{\varphi} - 3a^2\dot{a}\dot{\varphi} - a^3\frac{dV}{d\varphi} - \phi_\varphi,$$

and $\omega^a = da - \dot{a}dt$, $\omega^\varphi = d\varphi - \dot{\varphi}dt$ are contact 1-forms on $\mathbb{R} \times TM$.

Let ξ be a vector field on M , locally expressed by

$$\xi = \alpha(a, \varphi) \frac{\partial}{\partial a} + \beta(a, \varphi) \frac{\partial}{\partial \varphi}, \quad (75)$$

and let $J^2\xi$ be its second jet prolongation (see (28)),

$$J^2\xi = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \varphi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\varphi}} + \ddot{\alpha} \frac{\partial}{\partial \ddot{a}} + \ddot{\beta} \frac{\partial}{\partial \ddot{\varphi}}, \quad (76)$$

where

$$\begin{aligned} \dot{\alpha} &= \frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial \varphi} \dot{\varphi}, & \dot{\beta} &= \frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial \varphi} \dot{\varphi}, \\ \ddot{\alpha} &= \frac{\partial^2 \alpha}{\partial a^2} \dot{a}^2 + 2 \frac{\partial^2 \alpha}{\partial a \partial \varphi} \dot{a} \dot{\varphi} + \frac{\partial^2 \alpha}{\partial \varphi^2} \dot{\varphi}^2 + \frac{\partial \alpha}{\partial a} \ddot{a} + \frac{\partial \alpha}{\partial \varphi} \ddot{\varphi}, \\ \ddot{\beta} &= \frac{\partial^2 \beta}{\partial a^2} \dot{a}^2 + 2 \frac{\partial^2 \beta}{\partial a \partial \varphi} \dot{a} \dot{\varphi} + \frac{\partial^2 \beta}{\partial \varphi^2} \dot{\varphi}^2 + \frac{\partial \beta}{\partial a} \ddot{a} + \frac{\partial \beta}{\partial \varphi} \ddot{\varphi}. \end{aligned}$$

The following theorem is a reformulation of Theorem 8 for source form (74), associated with the Klein–Gordon Lagrangian and an external force; in this case, a part of necessary and sufficient conditions expressed by the Killing equation $\partial_\xi g = 0$ (31) can be integrated, hence the symmetry ξ is found explicitly.

Theorem 15. *Let ε (74) be a source form on $\mathbb{R} \times T^2M$, associated to the Klein–Gordon Lagrangian λ and an external force (ϕ_a, ϕ_φ) on TM . Let ξ be a vector field on M . The following two conditions are equivalent:*

- (a) *A vector field ξ on M is a generator of invariance transformations of ε .*
- (b) *ξ has an expression (75), where*

$$\alpha = A \frac{1}{\sqrt{a}} \left(\exp \left(\sqrt{\frac{3}{8}} \varphi \right) + C \exp \left(-\sqrt{\frac{3}{8}} \varphi \right) \right), \quad (77)$$

$$\beta = -\sqrt{6}A \frac{1}{a\sqrt{a}} \left(\exp \left(\sqrt{\frac{3}{8}} \varphi \right) - C \exp \left(-\sqrt{\frac{3}{8}} \varphi \right) \right) + B. \quad (78)$$

Here, $A, B \in \mathbb{R}$, and $C > 0$ are real numbers, and ε has an expression (74), where potential function V and forces ϕ_a, ϕ_φ satisfy the equations

$$\begin{aligned} \frac{9}{2} \alpha a V + 3 \beta a^2 \frac{dV}{d\varphi} + 6 \frac{\partial \alpha}{\partial \varphi} a \frac{dV}{d\varphi} - \frac{\partial}{\partial a} (\alpha f_a) - \frac{\partial \beta}{\partial a} f_\varphi - \beta \frac{\partial f_a}{\partial \varphi} \\ - \frac{\partial \phi_a}{\partial \dot{a}} \left(\frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial \varphi} \dot{\varphi} \right) - \frac{\partial \phi_a}{\partial \dot{\varphi}} \left(\frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial \varphi} \dot{\varphi} \right) = 0, \end{aligned}$$

and

$$\begin{aligned} & \frac{3}{2}\alpha a^2 \frac{dV}{d\varphi} + \beta a^3 \frac{d^2V}{d\varphi^2} + 3\frac{\partial\alpha}{\partial\varphi} a^2 V - \frac{\partial}{\partial\varphi} (\beta f_\varphi) - \frac{\partial\alpha}{\partial\varphi} f_a - \alpha \frac{\partial f_\varphi}{\partial a} \\ & - \frac{\partial\phi_\varphi}{\partial\dot{a}} \left(\frac{\partial\alpha}{\partial a} \dot{a} + \frac{\partial\alpha}{\partial\varphi} \dot{\varphi} \right) - \frac{\partial\phi_\varphi}{\partial\dot{\varphi}} \left(\frac{\partial\beta}{\partial a} \dot{a} + \frac{\partial\beta}{\partial\varphi} \dot{\varphi} \right) = 0. \end{aligned}$$

Proof. Computing the Lie derivative of ε (74) with respect to $J^2\xi$, we obtain, in a straightforward way, that the Noether–Bessel–Hagen equation is equivalent to the following system of conditions:

$$\alpha + 2a \frac{\partial\alpha}{\partial a} = 0, \quad \frac{\partial\beta}{\partial a} a^2 - 6 \frac{\partial\alpha}{\partial\varphi} = 0, \quad 3\alpha + 2 \frac{\partial\beta}{\partial\varphi} a = 0. \quad (79)$$

Equations in (79) for components α, β of vector field ξ can be integrated. From the first equation of (79), we get

$$\alpha = c \frac{1}{\sqrt{a}} \exp(h(\varphi))$$

for some function $h = h(\varphi)$ and some $c \in \mathbb{R}$. Differentiating α , we have

$$\begin{aligned} \frac{\partial\alpha}{\partial\varphi} &= \alpha \frac{dh}{d\varphi}, \quad \frac{\partial^2\alpha}{\partial\varphi^2} = \alpha \left(\frac{dh}{d\varphi} \frac{dh}{d\varphi} + \frac{d^2h}{d\varphi^2} \right), \\ \frac{\partial\alpha}{\partial a} &= -\frac{1}{2a} \alpha, \quad \frac{\partial^2\alpha}{\partial a \partial\varphi} = -\frac{1}{2a} \alpha \frac{dh}{d\varphi}, \end{aligned}$$

and, from the second and third equations of (79), we get

$$\frac{\partial^2\beta}{\partial a \partial\varphi} = \frac{6}{a^2} \alpha \left(\frac{dh}{d\varphi} \frac{dh}{d\varphi} + \frac{d^2h}{d\varphi^2} \right) = \frac{9}{4a^2} \alpha = \frac{\partial^2\beta}{\partial\varphi \partial a},$$

hence we get a second-order ordinary equation for $h = h(\varphi)$,

$$\frac{d^2h}{d\varphi^2} + \frac{dh}{d\varphi} \frac{dh}{d\varphi} = \frac{3}{8},$$

which is directly integrable with the solution

$$h(\varphi) = \ln \left(\exp \left(\sqrt{\frac{3}{2}} \varphi \right) + C \right) - \sqrt{\frac{3}{8}} \varphi + \text{const.} \quad (80)$$

for some real numbers $C > 0$, $c_0 \in \mathbb{R}$. Thus, α becomes of the form (77), where $A = c \exp(c_0)$.

The third equation of (79) now reads

$$\frac{\partial\beta}{\partial\varphi} = -\frac{3}{2a} A \frac{1}{\sqrt{a}} \left(\exp \left(\sqrt{\frac{3}{8}} \varphi \right) + C \exp \left(-\sqrt{\frac{3}{8}} \varphi \right) \right),$$

which can be directly integrated, hence

$$\beta = -\sqrt{6}A \frac{1}{a\sqrt{a}} \left(\exp \left(\sqrt{\frac{3}{8}}\varphi \right) - C \exp \left(-\sqrt{\frac{3}{8}}\varphi \right) \right) + p(a)$$

for some function $p = p(a)$. Substituting this form of β into the second equation of (79), we easily observe that p must be constant, $p(a) = B \in \mathbb{R}$, that is, β is given by (78). The pair α, β is the unique solution of system (79). Note that components of α (77), β (78) of the generator of invariance transformations of ε are dependent,

$$\beta - B = -4 \frac{1}{a} \frac{dh}{d\varphi} \alpha,$$

where $h = h(\varphi)$ is given by (80). □

In the next propositions, we analyze necessary and sufficient conditions for potential function V such that ξ is a symmetry of Lagrangian λ , and conditions for V and external forces (ϕ_a, ϕ_φ) such that vector field ξ is a symmetry of source form ε . Moreover, *exact solutions* of equations of motion (68), (69), obeying energy condition (67), are studied.

Corollary 16. *Let λ be the Klein–Gordon Lagrangian, $\lambda = \mathcal{L}dt$ (64). A vector field ξ (75) on TM is a generator of invariance transformations of λ if and only if its coefficients α, β are of the form (77), (78), and one of the following conditions is satisfied:*

- (i) $\alpha = 0$ ($A = 0$), $\beta = B \neq 0$, and $V = V(\varphi)$ is constant. In this case, the equations of motion (68), (69) and the energy condition (67) have a solution for $V = 0$ only, namely the mapping $\gamma : \mathbb{R} \rightarrow TM$ given by

$$\begin{aligned} \gamma(t) &= (a(\gamma(t)), \varphi(\gamma(t))) \\ &= \left(k_1 \sqrt[3]{3t + k_2}, \pm \sqrt{\frac{2}{3}} \ln |3t + k_2| + k_3 \right), \quad k_1, k_2, k_3 \in \mathbb{R}. \end{aligned} \quad (81)$$

The Noether conserved current (Corollary 11, (48)) along extremals (solutions of (68), (69)) reads

$$L = g_{ij} \xi^i \dot{x}^j = -Ba^3 \dot{\varphi}.$$

- (ii) $\alpha \neq 0$ ($A \neq 0$), $B = 0$, and $V = k(\exp(\sqrt{\frac{3}{8}}\varphi) - C \exp(-\sqrt{\frac{3}{8}}\varphi))^2$, where $k > 0$. In this case, the equations of motion (68), (69) and the energy condition (67) have a solution $\gamma : \mathbb{R} \rightarrow TM$ given by

$$\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} - V(\varphi) = 0, \quad \ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + \frac{dV}{d\varphi} = 0, \quad 3 \frac{\dot{a}^2}{a^2} - \frac{1}{2} \dot{\varphi}^2 - V(\varphi) = 0.$$

The Noether conserved current (Corollary 11, (48)) along extremals (solutions of (68), (69)) reads

$$L = g_{ij}\xi^i\dot{x}^j = 6a\dot{a}\alpha.$$

Corollary 17. Let ε (74) be a source form on $\mathbb{R} \times T^2M$, associated to Klein–Gordon Lagrangian λ and an external force $\phi = (\phi_a, \phi_\varphi)$ on TM . A vector field ξ on M , expressed by (75), is a generator of invariance transformations of ε if and only if α, β are of the form (77), (78), respectively, and $V = V(\varphi)$ and (ϕ_a, ϕ_φ) satisfy the following:

Case I. $\phi_a = \phi_\varphi = 0$:

$$\frac{3}{2}\alpha V + \beta a \frac{dV}{d\varphi} + 2 \frac{\partial \alpha}{\partial \varphi} \frac{dV}{d\varphi} = 0, \quad (82)$$

$$\frac{3}{2}\alpha \frac{dV}{d\varphi} + \beta a \frac{d^2V}{d\varphi^2} + 3 \frac{\partial \alpha}{\partial \varphi} V = 0. \quad (83)$$

Nontrivial solutions of (82), (83), read

(i) $\alpha = 0$ ($A = 0$), $B \neq 0$, V linear in φ ;

(ii) $\alpha \neq 0$ ($A \neq 0$), $B = 0$, and

$$V = k \left(\exp \left(\sqrt{\frac{3}{8}} \varphi \right) - C \exp \left(-\sqrt{\frac{3}{8}} \varphi \right) \right)^2, \quad k > 0.$$

Equations of motion (68), (69) and the energy condition (67) have solutions described by Proposition 16. Moreover, if V is linear, i.e. $V = P\varphi + Q$, $P \neq 0$, then (68), (69), (67) possess a solution described by

$$\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} - P\varphi - Q = 0, \quad \ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + P = 0, \quad 3 \frac{\dot{a}^2}{a^2} - \frac{1}{2} \dot{\varphi}^2 - P\varphi - Q = 0.$$

Case II. ϕ_a, ϕ_φ are conservative, i.e. $f_a = 3a^2V(\varphi)$, $f_\varphi = a^3dV/d\varphi$:

$$V = V(\varphi) \text{ is arbitrary.}$$

Equations (68), (69), (67) possess a solution described by

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{2} \dot{\varphi}^2 = 0, \quad \ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} = 0, \quad V(\varphi) = 3 \frac{\dot{a}^2}{a^2} - \frac{1}{2} \dot{\varphi}^2.$$

Case III. ϕ_a, ϕ_φ depend on a, φ variables only:

$$\begin{aligned} & \alpha \left(\frac{9}{2} a V - 6a \frac{dh}{d\varphi} \frac{dV}{d\varphi} + \frac{1}{2a} f_a - \frac{\partial f_a}{\partial a} + 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial f_a}{\partial \varphi} - 6 \frac{1}{a^2} \frac{dh}{d\varphi} f_\varphi \right) \\ & + B \left(3a^2 \frac{dV}{d\varphi} - \frac{\partial f_a}{\partial \varphi} \right) = 0, \end{aligned} \quad (84)$$

$$\begin{aligned} \alpha \left(3 \frac{dh}{d\varphi} a^2 V + \frac{3}{2} a^2 \frac{dV}{d\varphi} - 4 \frac{dh}{d\varphi} a^2 \frac{d^2 V}{d\varphi^2} - \frac{dh}{d\varphi} f_a + \frac{3}{2a} f_\varphi - \frac{\partial f_\varphi}{\partial a} + 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial f_\varphi}{\partial \varphi} \right) \\ + B \left(a^3 \frac{d^2 V}{d\varphi^2} - \frac{\partial f_\varphi}{\partial \varphi} \right) = 0, \end{aligned} \quad (85)$$

where $B \in \mathbb{R}$, and $h = h(\varphi)$ is function given by (80).

A nontrivial solution of (84), (85), read

$$\alpha = 0 (A = 0), \quad \beta = B \in \mathbb{R}, \quad f_a = 3a^2 V, \quad f_\varphi = a^3 \frac{dV}{d\varphi},$$

where $V = V(\varphi)$ is arbitrary.

Case IV. ϕ_a, ϕ_φ are dissipative, i.e. $\phi_a = -\phi_{11}\dot{a} - \phi_{12}\dot{\varphi}$, $\phi_\varphi = -\phi_{12}\dot{a} - \phi_{22}\dot{\varphi}$ for some functions $\phi_{ij} : M \rightarrow \mathbb{R}$:

$$\begin{aligned} \alpha \left(\frac{9}{2} a V - 6a \frac{dh}{d\varphi} \frac{dV}{d\varphi} \right) + 3Ba^2 \frac{dV}{d\varphi} &= 0, \\ \alpha \left(3a^2 \frac{dh}{d\varphi} V + \frac{3}{2} a^2 \frac{dV}{d\varphi} - 4a^2 \frac{dh}{d\varphi} \frac{d^2 V}{d\varphi^2} \right) + Ba^3 \frac{d^2 V}{d\varphi^2} &= 0, \\ \alpha \left(-\frac{1}{a} \phi_{11} + 12 \frac{1}{a^2} \frac{dh}{d\varphi} \phi_{12} + \frac{\partial \phi_{11}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{11}}{\partial \varphi} \right) + B \frac{\partial \phi_{11}}{\partial \varphi} &= 0, \\ \alpha \left(\frac{dh}{d\varphi} \phi_{11} - 2 \frac{1}{a} \phi_{12} + 6 \frac{1}{a^2} \frac{dh}{d\varphi} \phi_{22} + \frac{\partial \phi_{12}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{12}}{\partial \varphi} \right) + B \frac{\partial \phi_{12}}{\partial \varphi} &= 0, \\ \alpha \left(\frac{dh}{d\varphi} \phi_{11} - 2 \frac{1}{a} \phi_{12} + 6 \frac{1}{a^2} \frac{dh}{d\varphi} \phi_{22} + \frac{\partial \phi_{12}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{12}}{\partial \varphi} \right) + B \frac{\partial \phi_{12}}{\partial \varphi} &= 0, \\ \alpha \left(2 \frac{dh}{d\varphi} \phi_{12} - 3 \frac{1}{a} \phi_{22} + \frac{\partial \phi_{22}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{22}}{\partial \varphi} \right) + B \frac{\partial \phi_{22}}{\partial \varphi} &= 0. \end{aligned}$$

Nontrivial solutions of this system read

(i) $\alpha = 0$ ($A = 0$), $B \neq 0$, V linear in φ , and force coefficients $\phi_{11}, \phi_{12}, \phi_{22}$ are functions of variable a only.

(ii) $\alpha \neq 0$ ($A \neq 0$), $B = 0$, $V = k(\exp(\sqrt{\frac{3}{8}}\varphi) - C \exp(-\sqrt{\frac{3}{8}}\varphi))^2$, $k > 0$, and $\phi_{11}, \phi_{12}, \phi_{22}$ satisfy

$$\begin{aligned} -\frac{1}{a} \phi_{11} + 12 \frac{1}{a^2} \frac{dh}{d\varphi} \phi_{12} + \frac{\partial \phi_{11}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{11}}{\partial \varphi} &= 0, \\ \frac{dh}{d\varphi} \phi_{11} - 2 \frac{1}{a} \phi_{12} + 6 \frac{1}{a^2} \frac{dh}{d\varphi} \phi_{22} + \frac{\partial \phi_{12}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{12}}{\partial \varphi} &= 0, \\ 2 \frac{dh}{d\varphi} \phi_{12} - 3 \frac{1}{a} \phi_{22} + \frac{\partial \phi_{22}}{\partial a} - 4 \frac{1}{a} \frac{dh}{d\varphi} \frac{\partial \phi_{22}}{\partial \varphi} &= 0, \end{aligned}$$

where $h = h(\varphi)$ is given by (80); this system obey for instance $\phi_{11} = a$, $\phi_{12} = 0$, $\phi_{22} = -\frac{1}{6}a^3$.

In conclusion, the existence of Noether–Bessel-Hagen symmetry selects the form of external force and self-interacting potential of the scalar field.

5. Discussion and Conclusions

Starting from the Noether–Bessel-Hagen theorem, it is possible to find out symmetries for dynamical systems and then reducing them. The approach is particularly useful, with respect to the Noether standard, for non-Lagrangian systems involving external forces or dissipative terms. In fact, the Noether–Bessel-Hagen symmetry can be directly searched for the equations of motion once the (1) condition is satisfied for the dynamical system.

In this work, we provide new analysis of the Noether–Bessel-Hagen equation for mechanical systems on $\mathbb{R} \times T^2M$ with external forces, and discuss necessary and sufficient conditions for these systems (namely with conservative, dissipative, or variational forces) to possess a Noether–Bessel-Hagen symmetry. Conserved Noether currents for systems with variational external forces are also discussed. After developing the formal structure of the theory, we focused on two main examples: a damped harmonic oscillator and a scalar–tensor cosmological model. In the former case, we recover, as conserved quantity the angular momentum, as expected.

In the latter example, we considered the field equations of a minimally coupled scalar–tensor model with unknown potential $V(\varphi)$ and external force terms. The approach leads to a system of differential equations which, once solved, according to the existence of a symmetry, provides the form of the scalar-field potential. We selected different shapes for $V(\varphi)$ showing that the space of solutions in this case, is different than the one obtained by applying the standard Noether Symmetry Approach to the cosmological Lagrangian.

We presented a new way to look for symmetry of dynamical systems considering, specifically, those systems which are not variational. In this sense, the Noether–Bessel-Hagen Symmetry Approach is a generalization of the Noether Symmetry Approach. In forthcoming papers, we will apply the method to classes of cosmological models derived from modified theories of gravity like those discussed in [56, 60].

Acknowledgments

ZU acknowledges support by the ESF in Science without borders project, reg. no. CZ.02.2.69/0.0./0.0./16_027/0008463 within the Operational Programme Research, Development and Education. FB and SC acknowledge the support of Istituto Nazionale di Fisica Nucleare (INFN) (iniziative specifiche MOONLIGHT2, GINGER and QGSKY). This paper is based upon work from COST action

CA15117 (CANTATA), supported by COST (European Cooperation in Science and Technology).

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