BENCHMARK TRACKING PORTFOLIO PROBLEMS WITH STOCHASTIC ORDERING CONSTRAINTS

Field of study: Finance

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Chapter 1

Introduction

The financial world is complex and cannot be easily understood. Getting out of it becomes a demanding effort also for the best of the practitioner since financial markets represent a thick and savage jungle for most of the people who work in the financial industry. Economists have given great attention to test the concept of market efficiency and rationality. However, agents have unstable and unpredictable preferences and they do not make rational choices with them\footnote{Financial literature assume agents with constant preferences. However, empirical evidences in the financial markets show the presence of contagious enthusiasm or worries among different kind of investors.}. For this reason, the problem of choice is still a challenging issue for the financial agents or investors since “it is often said that investment management is an art, not a science” (Fabozzi, 2012).

The XXI century seems to represent a clear empirical example of the complexity of financial markets and their imperfections. Since the introduction of the financial modelling in the 1980s to price the financial derivatives, the level of complexity of the markets has forcefully increased, generating the so called “Century of crisis”. In fact, the first decade of XXI century will always be remembered as the most dramatic economic and financial period in history. Three different crises marked these years: the Dot-com speculative Bubble in 2001-2002 (started in March 2000), the sub-prime mortgage crisis in 2007-2009 and the following Eurozone sovereign debt crisis and economic recession.

The crises which characterized the first decade of the XXI century dictate the feebleness of the modern financial models and instruments needed to describe the financial sector and its features. The research of new concepts and diverse approaches is therefore
essential to guarantee and safeguard the stability of the entire system. Moreover, financial crises are not blimps but incentives to improve the knowledge of the markets and the behavior of the agents. Crises represent an intrinsic feature of the financial world that also a perfect modelization should consider. For this reason, the research of new ways to build portfolio models is crucial to protect the investors’ wealth.

The concept of creating portfolios capable to maximize investor’s utility can be obtained optimizing different measures presented in the financial literature starting from the seminal work of Markowitz (1952). In particular, we observed the importance to mimic the behavior of a stock index or to replicate its returns during different phases of the financial cycle. This problem which aim to select the optimal portfolio composition in order to reduce the difference between its returns and the given benchmark ones is called benchmark tracking problem and it is part of a more general framework and area of research known as the Modern Portfolio Theory.
Chapter 2

Objective and Structure of Doctoral Dissertation

The main objective of this essay is to analyze the entire benchmark tracking problem. Facing this issue, portfolio managers want to find the optimal portfolio composition that maximizes the management style. In particular, benchmark tracking portfolio strategies could be divided into three main categories: passive, enhanced indexing and active. This essay addresses with these three problems proposing theoretical and methodological solution to maximize investors’ preferences. Empirical applications involving different phases of the financial cycle during the last decade enforce the goodness of the proposed methodology. The most common approach in passive portfolio management is index tracking. In this strategy, investors select portfolios that mimic the behavior of an index representing the entire market or one of its sector. To find the optimal combination, the definition and minimization of a distance measure between tracking portfolio and benchmark index is the crucial point to efficiently manage this problem. In contrast, active portfolio management tries to generate excess returns picking stocks which are expected to outperform the market and avoiding assets that are expected to underperform it. Both approaches have their advantages and disadvantages: active strategies rely heavily on superior predictions while passive strategies require few assumptions about future price movements. Passive strategies will also copy the benchmark’s poor behavior while active strategies can react more flexibly in bear markets; etc. In the middle we find the enhanced indexing strategies that try to capture the best feature
of both approaches proposing a portfolio composition that minimizes risk looking for extra-performances.

Thus, in Chapter 2 starting from the problem to mimic the performance of a financial index considering all its components or a subset only, we propose a new dispersion measure of the tracking error. This measure, called tracking error quantile regression, results to be suitable to track a given benchmark not only from a theoretical and but also from an empirical point of view. In fact, it overcomes some drawbacks of the common dispersion measures such as non-linearity and symmetry confirmed in an empirical application. The contribution made by Chapter 2 is theoretical and methodological since it describes the introduced dispersion measure based on the quantile regression with its theoretical structure. The methodological contribution is then developed proposing a realistic LP model to solve the enhanced index problem. This problem represents a hot topic of research since every portfolio manager aims not only to track an index from above reducing the dispersion measure but also to obtain gains in the out of sample analysis. For this reason, we introduce stochastic dominance constraints in the minimization problem of the tracking error to enhance the portfolio performances. An empirical application is also developed showing the enhancement of portfolios wealth path in the out of sample analyses.

Chapter 3 generalizes the concept of dispersion measure reviewing the class of the coherent expectation bounded risk measures for the benchmark tracking problem. These measures, like the class of Gini dispersion measures, represent a useful metric to improve the decisional problem in the replication of the performances of an given index. Then, we introduce the methodology of stochastic investment chain grounded on the concept to create portfolio with stronger behavior derived from three consequent optimization steps increasing the level of stochastic dominance where the dominant portfolio become the benchmark. The contribution of this chapter is theoretical and methodological. On one hand, we analyze the linearity of these measures proposing different portfolio problems based on the dispersion component of coherent expectation bounded risk measures. In particular, since this class of measures is consistent with Rothschild–Stiglitz ordering, we could derive a tracking error problem consistent with this ordering. On the other hand, we theoretically develop linear programming formulation to solve portfolio problems with bounded third order stochastic dominance constraints. In this framework, we considering an aggressive Rachev utility function which is consistent with the preference
of non-satiable nor risk seeking nor risk averse investors we develop the concept of stochastic investment chain.

Finally in Chapter 4, we deal with portfolio strategies for active management. In the Modern Portfolio Theory, the maximization of the investors’ future wealth is still an relevant problem. Thus, we propose portfolio strategy which does not focus on the risk minimization but on the maximization of performance measures considering different ratios. The contribution of this chapter is theoretical, methodological and empirical. Since investors maximize their utility in a reward-risk sense, we implement linear portfolio optimization problems maximizing four different performance measures. In the theoretical part, we review the linear programming model of two performance measures while we develop the theoretical formulation for the Sharpe Ratio and the Mean Absolute Semideviation Ratio. Then, introducing first and second order stochastic dominance constraints we propose different portfolio selection models to strengthen the performances of invested portfolios. Finally, we empirically test the benefit to introduce stochastic dominance in portfolio problems considering its impact in the maximization of future wealth.
Chapter 3

Content of Doctoral Dissertation

Abstract

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Chapter 4

Procedure of Doctoral Dissertation

4.1 Tracking Error Quantile Regression. A Dispersion Measure for the Benchmark Tracking Problem

One of the most important objectives that every fund manager has to achieve is the index tracking problem. Many portfolios are managed to a benchmark or index and they are expected to replicate, its returns (e.g., an index fund), while others are supposed to be “actively managed” deviating slightly from the index in order to generate active returns. The tracking problem has been broadly described in the financial literature from different points of views. On the one hand, the research community focuses on the identification of efficient algorithms to solve the optimization problem through the development of a large diversity of heuristics and metaheuristics (Angelelli et al., 2012; Beasley et al., 2003). On the other hand, several approaches have been introduced in order to describe empirical evidences, to improve the decisional problem or to propose different methodologies dealing with the index tracking problem (Krink et al., 2009; Maringer and Oyewumi, 2007).

Index tracking problems is related with a benchmark portfolio against which the performance of a managed one is compared. This comparison is based on the distribution of the active portfolio return, defined as the difference $X - Y$, in which $X = r\beta$ is the
random variable of the invested portfolio returns with weights represented by the vector $\beta$ while $Y$ denotes the benchmarks’ returns. Performance and risk of the portfolio managers’ strategies are based on this difference. In particular, a measure of performance of the invested portfolio relative to the benchmark is the average active return, also known as portfolio alpha, which is calculated as the difference in the sample means:

$$\hat{\alpha} = \mathbb{E}[X] - \mathbb{E}[Y]$$

(4.1)

Differently, a widely used risk measure of how close the portfolio returns are to the benchmark is a deviation measure of the active return, also known as tracking error (TE). These two measures are the decisional parameters in the problem of choice for portfolio managers. There are common measures (Mansini et al., 2003) are broadly used in the benchmark tracking but they present some intuitive drawbacks and theoretical lacks. For this reason, we consider the quantile regression method to build a dispersion measure of the tracking error suitable for the benchmark tracking problem.

**Definition 1**

Let $Y$ be a random variable of benchmark returns with realization $y_t$ for $t = 1, \ldots, T$, let $X = r\beta$ the returns of the invested portfolio of its $N$ components and $\tau$ be the quantile of interest such that $0 \leq \tau \leq 1$. Let $\varepsilon_t = \sum_{n=1}^{N} r_{t,n}\beta_n - y_t$ the difference between portfolio and benchmark returns at time $t$, we define the **tracking error quantile regression (TEQR)** at given $\tau$ as:

$$\text{TEQR} \quad \sigma(\varepsilon|\tau) = \tau \sum_{t=1}^{T} \varepsilon_t 1_{[\varepsilon_t \geq 0]} + (1 - \tau) \sum_{t=1}^{T} \varepsilon_t 1_{[\varepsilon_t < 0]}$$

(4.2)

In (4.2), the first term is the sum of the positive residuals while the second term is the sum of negative residuals. The first one represents the observations that lie above the regression line and they receive a weight of $\tau$, while the second are the observations that lie below the regression line and they receive a weight of $(1 - \tau)$.

As underlined by Koenker and Bassett (1978), the quantile regression problem does not present a close form solution as the mean square error but it is the result of a minimization problem. Let $u$ and $\nu$ two slack variables such that:

$$u_t = \varepsilon_t 1_{[\varepsilon_t \geq 0]} \quad \forall t = 1, \ldots, T$$

$$\nu_i = \varepsilon_i 1_{[\varepsilon_i < 0]} \quad \forall i = t, \ldots, T$$

(4.3)
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It is possible to express the quantile regression as a solution of the following minimization problem:

$$\min_{(w,u,\nu) \in \mathbb{R}^p \times \mathbb{R}^{2n}_+} \left\{ \tau \mathbf{1}^T u + (1 - \tau) \mathbf{1}^T \nu \mid r \beta - u + \nu = y \right\}$$  

(4.4)

The linearity of the formulation and its theoretical support make of the tracking error quantile regression a suitable dispersion measure to solve the benchmark tracking portfolio problem. This measure is defined fixing the value of $\tau$. As we discuss later, the choice of the quantile represents an interesting topic of research and the possibility to switch its value during the time allows to capture the features of the financial markets.

Thus, let the log-return of equity index $Y$ s.t. $y_t, t = 1, \ldots, T$ and of its $N$ components being $r_1, r_2, \ldots, r_N$. We define the slack variable $u$ and $\nu$ (4.3).

The linear programming benchmark tracking problem with the tracking error quantile regression dispersion measure is:

$$\min_{\beta, u, \nu} \sum_{t=1}^{T} \tau u_t + (1 - \tau) \nu_t$$

s.t. $r_t \beta - u_t + \nu_t = y_t \quad \forall t = 1, \ldots, T$

$$\sum_{n=1}^{N} \beta_n = 1$$

$$\mathbb{E}[X] - \mathbb{E}[Y] \geq K^*$$

$$u_t, \nu_t \geq 0 \quad \forall t = 1, \ldots, T$$

$$lb \leq \beta_n \leq ub \quad \forall n = 1, \ldots, N$$

(4.5)

Then, we propose a realistic formulation to solve the enhanced index benchmark tracking problem. This complete formulation takes into account a linear penalty objective function and buy-in threshold level to reduce the portfolio turnover and risk management duties. Moreover, whether the introduction of stochastic dominance constraints enhances the benchmark tracking model, its formulation strongly increases the dimensionality and the computational complexity of the problem. In particular, we consider the linearizations proposed by Kopa (2010) and Kuosmanen (2004).

The enhanced index benchmark tracking problem is solved considering the minimization of a dispersion measure of the tracking error, the tracking error quantile regression (4.2), which could be formulated as linear (4.4), and the minimization of the transaction costs.
To enhance the performance in the risk minimization, we introduce first and second order stochastic dominance constraints. Let the log-return of equity index $Y$ with realization $y_t$, $t = 1, \ldots, T$ and of its $N$ components being $R = r_1, r_2, \ldots, r_N$. The tracking error $\varepsilon_t = \sum_{n=1}^{N} r_{t,n} \beta_n - y_t$ is minimized considering the tracking error quantile regression. Then, let $tc^+$ and $tc^-$ the transaction costs to the buying and selling portfolios $\omega^+$ and $\omega^-$ with a buy-in threshold level $\theta$. We define the enhanced indexation benchmark tracking problem with stochastic dominance constraints as:

$$\min_{\beta,u,\nu,\omega^+,\omega^-} \sum_{t=1}^{T} \tau u_t + (1 - \tau) \nu_t + tc^+ \omega^+ + tc^- \omega^-$$

s.t. $$r_t \beta - u_t + \nu_t = y_t \quad \forall t = 1, \ldots, T$$

$$\sum_{n=1}^{N} \beta_n = 1$$

$$\mathbb{E}[X] - \mathbb{E}[Y] \geq K^*$$

$$\omega^+_n - \omega^-_n = \beta_n - \beta^{old}_n \quad \forall n = 1, \ldots, N$$

$$\sum_{n=1}^{N} |\beta_n - \beta^{old}_n| \leq \theta \quad \forall n = 1, \ldots, N$$

$$X \succeq_{FSD} Y \text{ or } X \succeq_{SSD} Y$$

$$lb \leq \omega^+_n, \omega^-_n, \beta_n \leq ub \quad \forall n = 1, \ldots, N$$

$$u_t, \nu_t \geq 0 \quad \forall t = 1, \ldots, T$$

(4.6)

The choice of the stochastic dominance degree defines the strength of the enhancing process. In particular, the solution with FSD constraints is the portfolio chosen by all non-satiable investors while the SSD solution leads to portfolio which is optimal for non-satiable risk averse investors. Finally, if the first portfolio is a linear mixed-integer problem depending from the length of the time series data, the second problem could be efficiently solved in a linear way.
4.2 Dispersion Measure for the Benchmark Tracking Portfolio Problem and Third Order Stochastic Dominance Constraints

In Chapter 3, we classify the introduced measure of dispersion of the tracking error in a more general framework considering the class of the coherent expectation bounded risk measures. Linking the concept of risk measure with respect to a given orderings, we derive other linear measures suitable to solve the benchmark tracking problem. Then, we theoretically generalize the introduction of third order stochastic dominance in the portfolio problem introducing a linear formulation for its constraints. In this way, we obtain portfolio chosen by non-satiable risk averse investors with positive skewness. This concept allow to build portfolio which dominates in the third order sense an optimal portfolio with respect to the second order maximizing a performance measure that is consistent with none of the previous orders. Empirically speaking, this contribution is focused on the concept of stochastic investment chain which are grounded on a three step portfolio optimization problem with different orders of stochastic dominance.

Let us to introduce the class of concentration $L^p$-metrics. Considering the class of compound metrics we propose a concentration measure which could be applied to the benchmark tracking portfolio problem. Following Ortobelli et al. (2013), we say that $X$ is preferred to $Y$ with respect to the $\mu$-compound distance from $Z$ (namely, $X \succeq Y$) if and only if there exists a probability functional $\rho : \Lambda \times Z \times B \rightarrow R$ dependent on $\mu$ such that for any $t \in B$ and $X,Y \in \Lambda$, $\rho_X(t) \leq \rho_Y(t)$.

Considering the $L^p$ average compound metric, for every $p \geq 0$ we recall the $L^p$-metrics:

$$\mu_p(X,Y) = \mathbb{E}[[X - Y]^p]^{\min(1,1/p)};$$

the associated concentration measures are $\mu_{I,p}(X) = \mathbb{E}[[X - X_1]^p]^{\min(1,1/p)}$, where $X_1$ is an i.i.d. copy of $X$; and the associated dispersion measures are the central moments $\mu_{E[X],p}(X) = \mathbb{E}[[X - \mathbb{E}[X]]^p]^{\min(1,1/p)}$. The dispersion and concentration measures $\mu_{E[X],p}(X)$ and $\mu_{I,p}(X)$ are variability measures consistent with the $(p+1)$ Rothschild–Stiglitz order for any $p \geq 1$. We can consider for $L^p$ metrics the tracking-error measures

$$\rho_{X,p}(t) = (\mu_p(X1_{|X-Z| \geq t}, Z1_{|X-Z| \geq t}))^{\min(1,1/p)} - t^{p}P(|X-Z| \geq t)$$

$$= \mathbb{E}[(|X - Z|^p - q^p)_+] (4.7)$$
for any $t \in [0, +\infty)$ associated with a $\mu_p$ tracking error ordering. Moreover, $\rho_{X,p} = \rho_{Y,p}$ implies that $F|_{X-Z} = F|_{Y-Z}$.

To evaluate the importance of the introduction of stochastic dominance constraints in the benchmarking portfolio problem we propose three realistic models which takes into account a penalty function with transaction costs and a turnover threshold. Let $X = r\beta$ a random variable of the portfolio returns and $Y$ the random variable of the benchmark returns with realizations $r_t\beta$ and $y_t$ for $t = 1, \ldots, T$. Let $\beta \in \mathbb{S} = \{\beta \in \mathbb{R}^n | \sum_{n=1}^N \beta_n = 1\}$ and $q$ be a parameter such that $q \in \mathbb{Q} = \{\delta, \max_\beta(\max_t [r_t\beta, y_t])\}$ with $\delta$ small enough. Let $tc^+$ and $tc^-$ the transaction costs to buy and sell new securities and $\beta_{old}$ the composition of the invested portfolio before the optimization step. Then, the optimal portfolio composition which solve the benchmark tracking problem is obtained fixing the value of $q$ and solving the following linear programming problem:

$$
\min_{\beta, u, \nu, \omega^+, \omega^-} \frac{1}{T} \sum_{t=1}^T u_t + tc^+ \omega^+ + tc^- \omega^-
$$

s.t.

$$
u_t \geq r_t\beta - y_y \quad \forall t = 1, \ldots, T$$

$$
u_t \geq y_y - r_t\beta \quad \forall t = 1, \ldots, T$$

$$
u_t \geq \nu_t - q \quad \forall t = 1, \ldots, T$$

$$
\sum_{n=0}^N \beta_n = 1
$$

$$
E[X] - E[Y] \geq K^*
$$

$$
\omega_n^+ - \omega_n^- = \beta_n - \beta_{old}^n \quad \forall n = 1, \ldots, N
$$

$$
\sum_n |\beta_n - \beta_{old}^n| \leq \theta \quad n = 1, \ldots, N
$$

$$
(X \succeq_{FSD} Y \text{ or } X \succeq_{SSD} Y)
$$

$$
lb \leq \beta_n \leq ub \quad \forall n = 1, \ldots, N$$

$$
u_t \geq 0; \quad \nu_t \in \mathbb{R} \quad \forall t = 1, \ldots, T$$

where $u, \nu$ are two variable to linearize the associate benchmark tracking measure of the $L^p$ metric and $\omega^+, \omega^-$ two slack variables of the portfolio changes. The introduction of stochastic dominance constraints leads to built portfolios optimality different from investors’ preferences.
Financial agents and investors behave differently in their approach to the financial markets. For years several studies propose utility functions to describe their behavior and the preferences trying to draw a complete picture of the entire universe. Here, we develop linear formulation problems to maximize one kind of investors’ preferences represented by the Rachev utility functions (Rachev et al., 2008). It reflects the behavior of non-satiable nor risk averse nor risk seeking investors. In particular, maximizing this kind of preference, we solve the following portfolio selection problem. Let \( X = r\beta \) be a random variable of the portfolio returns and let \( \alpha_1 \) and \( \alpha_2 \) be two confidential levels. The Rachev utility is defined as the difference between the Conditional Value at Risk of the two sides of the returns distribution: 
\[
bCVaR_{\alpha_1}(-X) - aCVaR_{\alpha_2}(X)
\]
where \( a, b \in \mathbb{R}^+ \) are two positive coefficients.

**Proposition 1**

The solution of the maximization of the Rachev utility function with linear constraints is defined with the following mixed-integer linear programming portfolio selection problem:

\[
\max_{\beta, g, \lambda, \gamma, d} \quad \frac{1}{[\alpha_1 T]} \sum_{t=1}^{T} g_t - \frac{a}{b} \left( \gamma + \frac{1}{\alpha_2 T} \sum_{t=1}^{T} d_t \right)
\]

s.t.

\[
\sum_{n=1}^{N} \beta_n = 1
\]

\[
g_t \leq B\lambda_t \quad \forall t = 1, \ldots, T
\]

\[
g_t \geq r_t\beta - B(1 - \lambda_t) \quad \forall t = 1, \ldots, T
\]

\[
g_t \leq r_t\beta + B(1 - \lambda_t) - r_t\beta - \gamma \leq d_t \quad \forall t = 1, \ldots, T
\]

\[
lb \leq \beta_n \leq ub \quad \forall n = 1, \ldots, N
\]

\[
\lambda_t \in \{0, 1\}, \quad \gamma \in \mathbb{R} \quad \forall t = 1, \ldots, T
\]

\[
g_t \geq 0, \quad d_t \geq 0 \quad \forall t = 1, \ldots, T
\]

where \( \lambda \) is the binary variable.

The introduction of third order stochastic dominance relates on the possibility to build portfolio which dominates a benchmark which is optimal in the second order stochastic sense. Considering the maximization of an aggressive function such as the Rachev utility it is possible to construct portfolio with a suitable behavior for the investors. To
express the concept of the third order stochastic dominance constraints in a linear formu-
lization, we propose an approach based on three main works presented in the literature

In particular, whether Dentcheva and Ruszczyński (2003) propose a linear formulation
for the second order stochastic dominance while Hanoch and Levy (1970) introduce some
condition to satisfy second order stochastic dominance which can be extended to other
higher order.

**Theorem 1**

Let \( X \) and \( Y \) be two random variables with finite number of discrete realizations \( x_t \) and \( y_t \) for \( t = 1, \ldots, T \). Whether exists a \( \bar{t} \) such that

\[
F^{(2)}_X(t) \leq F^{(2)}_Y(t) \quad \forall t \leq \bar{t} \\
F^{(2)}_X(t) \geq F^{(2)}_Y(t) \quad \forall t > \bar{t} \\
E[X] > E[Y]
\]

(4.10)

then, \( X \succeq 3 Y \) and \( X \not\succeq 2 Y \).

This theorem can be used even in linear portfolio selection framework when instead of
classic third stochastic dominance we consider the third bounded stochastic dominance
order. As a matter of fact, Fishburn (1980) has shown that for any order greater than
the second one, the classic stochastic dominance order implies the bounded one but the
contrary is not always true.

**Proposition 2**

Assuming to consider two random variables \( X \) and \( Y \) that have discrete distributions
with realization \( r_i \beta \) and \( y_i \) for \( t = 1, \ldots, T \) which belong to the joint support \( U = \{ u_{(1)} \leq \cdots \leq u_{(j)} \leq \cdots \leq u_{(J)} \} \) where \( J \leq 2T \). Then, \( X \succeq Y \) (i.e., \( X \) dominates \( Y \) in
the bounded third order stochastic dominance sense) if there exists a \( i \in U \) such that:

\[
F^{(2)}_X(u_j) \leq F^{(2)}_Y(u_j) \quad \forall u_j \in U \leq \bar{i} \\
F^{(2)}_X(u_j) \geq F^{(2)}_Y(u_j) \quad \forall u_j \in U > \bar{i} \\
E[X] > E[Y]
\]

(4.11)
We generally know that we cannot consider the ui of the support of all portfolio problem but we have a good approx when the element of the number of the support are very large.

The concept of stochastic investment chain allows investors and portfolio manager to create aggressive strategies improving the reward-risk combination. For this reason, we propose an investment strategy which is based on a three steps portfolio optimization process when we introduce a new stochastic order to dominate previous dominant portfolio.

Let $A$, $B$ and $C$ be three steps which the related optimal portfolios (i.e. $X^A = r^A$) and let $Y$ be a random variable of the index benchmark returns. Considering the Rachev utility function we follow the following portfolio selection rule to build the stochastic investment chain:

A) $X^A$ FSD $Y$;  
B) $X^B$ SSD $X^A$;  
C) $X^C$ TSD $X^B$.

At each investment step, we evaluate the risk and the reward of the strategy to consider the advantages and disadvantages of this approach.

4.3 Linear Programming Active Management Strategy. The Maximization of Performance Measures

The active benchmark tracking portfolio problem is a investment strategy which aims to exceed the performance of a selected target benchmark and it is sometimes referred to as active portfolio management (Sharpe, 1994). It is well known that many professional investors achieve this benchmarking strategy: for instance, bond funds try to beat the Barclays Bond Index, commodity funds seek to beat the Goldman Sachs Commodity Index while several mutual funds take the Standard and Poors (S&P) 500 Index as their benchmark. The aim of Chapter 4 is to solve the benchmark tracking problem
implementing active strategies to manage portfolio which outperforms the benchmark index.

In the active strategy framework, the goal of portfolio managers is to maximize their future or final wealth considering different reward/risk investors’ profiles. In particular, maximizing future investors wealth, we generally use a reward/risk portfolio selection model applied either to historical series or to simulated scenario models (see, among others, Rachev et al., 2008 and Biglova et al., 2004). Let \( Y \) be the random variable representing the return of a given benchmark with realization \( y_t \) at time \( t \) for \( t = 1, \ldots, T \) composed by \( N \) assets with returns \( R = [r_1, \ldots, r_N]' \). Thus, the vector of the returns of an invested portfolio is defined by a random variable \( X \) such that \( X = r\beta \) with realization \( x_t = \sum_{n=1}^{N} r_{n,t}\beta_n \) and the tracking error is a random variable \( Z \) such that \( Z = X - Y \). To maximize the performances of a portfolio in the reward/risk framework, we provide the maximum expected reward \( \mu \) per unit of risk \( \rho \). This optimal portfolio is commonly called the market portfolio and it can be obtained with several possible reward/risk performance ratios (Cogneau and Hubner, 2009a-2009b) defined as:

\[
G(Z) = \frac{\mu(Z)}{\rho(Z)}
\]  

(4.12)

Here, we review and present four measure of performances given by the ratios between a reward and a risk measure.

The Sharpe ratio is a commonly used measure of portfolio performance. According to the Markowitz mean-variance analysis, Sharpe (1994) suggests that investors should maximize what is now referred to as the Sharpe Ratio (SR) given by:

\[
SR(Z) = \frac{\mathbb{E}[Z]}{\text{STD}(Z)}
\]  

(4.13)

where the numerator is the expected value and the denominator represent the standard deviation of excess returns.

The Rachev Ratio Biglova et al. (2004) is based on tail measures and it is isotonic with the preferences of non-satiable investors that are neither risk averse nor risk seekers. The Rachev Ratio (RR) is the ratio between the average of earnings and the mean of losses; that

\[
RR(Z, \alpha_1, \alpha_2) = \frac{\text{CVaR}_{\alpha_2}(-Z)}{\text{CVaR}_{\alpha_1}(Z)}
\]  

(4.14)
where the Conditional Value-at-Risk (CVaR), is a coherent risk measures (Rockafellar and Uryasev, 2002; Artzner et al., 1999).

In 2005, Martin et al. (2005) introduce a different reward risk measure: the Stable Tail Adjusted Return Ratio (STARR\(_\alpha\)). This measure is a generalization of the Sharpe Ratio but it allows to overcome the drawbacks of the standard deviation as a risk measure (Artzner et al., 1999). Thus, the STARR at the confidence level \(\alpha\) is expressed as:

\[
\text{STARR}(Z, \alpha) = \frac{E[Z]}{\text{CVaR}\_\alpha(Z)}
\] (4.15)

The STARR differently from the Sharpe Ratio considers a coherent risk measure and not a deviation one as risk sources.

Finally, we introduce a performance measure splitting the two components of the quantile regression dispersion measure. This ratio is based on the idea to divide positive and negative difference between the returns of the invested and benchmark portfolios and evaluate their mean in the absolute sense.

**Definition 2**

Let \(Z\) be a random variable with realization at time \(t\) equal to \(\varepsilon_t\) which represent the difference between the returns of the invested portfolio and the benchmark (i.e. \(\varepsilon_t = r_t \beta - y_t\)) and let \(u_t = (r_t \beta - y_t)\mathbb{I}_{[r_t \beta \geq y_t]}\) and \(\nu_t = |(y_t - r_t \beta)\mathbb{I}_{[r_t \beta < y_t]}|\) be two positive variables representing the two sides of the excess returns between investing and benchmark portfolios. We define the **Mean Absolute Semideviation Ratio** (MASDR) as:

\[
\text{MASDR}(Z) = \frac{E[Z\mathbb{I}_{[Z \geq 0]}]}{E[Z\mathbb{I}_{[Z < 0]}]} = \frac{1}{T} \sum_{t=1}^{T} u_t = \frac{1}{T} \sum_{t=1}^{T} \nu_t = \frac{E[\max(r_t \beta - y_t, 0)]}{E[\max(y_t - r_t \beta, 0)]}
\] (4.16)

The main problem to address in the active strategy framework is related with the linearization of the four performance measures presented before solving the following portfolio selection problem. Let \(X = r \beta\) be the return of the invested portfolio with realization \(x_t = r_t \beta\) at time \(t\) where \(r_t\) is the raw of the asset returns and let \(Y\) be the returns of the benchmark with realization \(y_t\). We define the common portfolio selection
problem as follow:

$$\max_{\beta} \frac{\mu(X - Y)}{\rho(X - Y)}$$

subject to

$$\sum_{n=1}^{N} \beta_n = 1$$

$$lb \leq \beta_n \leq ub \quad \forall n = 1, \ldots, N$$

(4.17)

where $\beta_n$ for $n = 1, \ldots, N$ is the portfolio weight vector and optimal solution of the minimization problem, $lb$ the lower bound and $ub$ the upper bound as maximum amount invested in a given asset. We could notice that the objective function is non linear since in the ratio the variable $\beta$ appears both to the numerator and to the denominator. For this reason analyzing the nature of different reward and risk measure, we linearize these objective functions following the theoretical structure in Stoyanov et al. (2007).

Applying the linearization technique developed in Stoyanov et al. (2007), we could reformulate the problem (4.17) where the objective function is represented by the STARR$_{\alpha}(X)$ (4.15). Let $\beta = \frac{w}{g}$ and $X^* = rw$ we obtain the following LP portfolio selection problem:

$$\min_{w,g,d,\gamma} \gamma + \frac{1}{\alpha T} \sum_{t=1}^{T} d_t$$

subject to

$$E[X^*] - gE[Y] = 1$$

$$-r_t w + gy_t - \gamma \leq d_t \quad \forall t = 1, \ldots, T$$

$$\sum_{n=1}^{N} w_n = g$$

$$g \, lb \leq w_n \leq g \, ub \quad \forall n = 1, \ldots, N$$

$$g \geq 0, \quad d_t \geq 0 \quad \forall t = 1, \ldots, T$$

(4.18)

where $\alpha$ is the confident level.

Differently, there are reward-risk ratios suggested in literature that are not in the class of the quasi-concave functions because both the numerator and the denominator are convex. However, Stoyanov et al. (2007) propose a mixed-integer linear programming (MILP) formulation for the Rachev Ratio introducing binary variables $\lambda_t, \forall t = 1, \ldots, T$ and the threshold $B$ such that $B \geq |x_t w|, \forall t = 1, \ldots, T$. 
In this case, setting an extremely high value of $B$ we could solve the following MILP problem which maximizes the $\text{RR}_{\alpha_1, \alpha_2}(X)$:

$$
\begin{align*}
\min_{w, g, f, d, \lambda, \gamma} & \quad - \frac{1}{|\alpha_2 T|} \sum_{t=1}^{T} f_t \\
\text{s.t.} & \quad f_t \leq B \lambda_t \quad \forall t = 1, \ldots, T \\
& \quad f_t \geq r_t w - B(1 - \lambda_t) \quad \forall t = 1, \ldots, T \\
& \quad f_t \leq r_t w + B(1 - \lambda_t) \quad \forall t = 1, \ldots, T \\
& \quad \sum_{t=1}^{T} \lambda_t = \lceil \alpha_2 T \rceil \\
& \quad \gamma + \frac{1}{|\alpha_1 T|} \sum_{t=1}^{T} d_t \leq 1 \\
& \quad \gamma + \frac{1}{|\alpha_1 T|} \sum_{t=1}^{T} d_t \leq 1 \\
& \quad \gamma \geq 0, \; d_t \geq 0, \; f_t \geq 0, \; \lambda_t \in [0, 1] \quad \forall t = 1, \ldots, T \\
\end{align*}
$$

when the real portfolio weights are obtained dividing the vector $w$ for the scalar $g$ such that $\beta = \frac{w}{g}$ and $\lceil \alpha T \rceil$ is the ceiling integer number of $\alpha T$.

Then, we propose linear formulations to maximize the common portfolio problem (4.17) when the measure of performance is the Sharpe Ratio (4.13) or the Mean Absolute Semideviation Ratio (4.16).

**Proposition 3**

The maximization of the Sharpe Ratio should be solve in a linear programming formulation considering that the reward measure satisfies the positive homogeneity property and it is concave while the risk measure is positive homogeneous and sub-addictive. Considering the fractional integral theory, we maximize the Sharpe Ratio solving the following
problem:

$$\min_{w,v,u,g} \sum_{i=1}^{M-1} \frac{1}{T} \sum_{k=1}^{T} v_{k,i} + u_{k,i}$$

s.t.

$$\mathbb{E}[X^*] - g \mathbb{E}[Y] = 1$$

$$\sum_{n=1}^{N} w_n = g$$

$$v_{k,i} \geq c + \frac{i}{M} (\mathbb{E}[X^*] - g \mathbb{E}[Y] - c) - r_k w + gy_k \quad \forall k = 1, \ldots, T, \ i = 1, \ldots, M$$

$$u_{k,i} \geq c + \frac{i}{M} (-\mathbb{E}[X^*] - g \mathbb{E}[Y] - c) + r_k w - gy_k \quad \forall k = 1, \ldots, T, \ i = 1, \ldots, M$$

$$g \text{lb} \leq w_n \leq g \text{ub} \quad \forall n = 1, \ldots, N$$

$$g \geq 0, \ u_{k,i} \geq 0, \ v_{k,i} \geq 0 \quad \forall k = 1, \ldots, T, \ i = 1, \ldots, M$$

(4.20)

where $M$ is a large integer and $c = -\max(\min_{\beta,\min} (r_t \beta - y_t), \max_{\beta,\max} (r_t \beta - y_t))$.

Thus, the optimal portfolio weight $\beta_n = \frac{w_n}{g}$ for $n = 1, \ldots, n$.

**Proposition 4**

The general performance measure optimization problem (4.17) is equivalent to the following linear programming problem:

$$\min_{w,d,g} \sum_{t=1}^{T} d_t$$

s.t.

$$\mathbb{E}[X^*] - g \mathbb{E}[Y] = 1$$

$$\sum_{n=1}^{N} w_n = g$$

$$d_t \geq gy_t - r_t \quad \forall t = 1, \ldots, T$$

$$g \text{lb} \leq w_n \leq g \text{ub} \quad \forall n = 1, \ldots, N$$

$$g \geq 0, \ d_t \geq 0 \quad \forall t = 1, \ldots, T$$

(4.21)

where the optimal portfolio composition $\beta = w/g$.

In an empirical application, we introduce the linear formulation of the first and second orders stochastic dominance constraints to increase the wealth of the invested portfolio keeping the optimization problem linear or mixed-integer linear programming.
The empirical analysis is based on three stock indexes: Russell 1000, S&P 500 and Nasdaq 100. In this essay, we report the significant main results of the entire work presented in the thesis. The analyzed time period covers the last decade from 31st December 2002 to 31st December 2013 and we propose investment strategies with monthly recalibration (20 days) with a total number of 125 optimization steps. We generally consider an historical moving window of 260 observations which is reduced to 120 time series data when we apply enhanced indexing strategies with first order stochastic dominance constraints. Every investment portfolio strategy starts on 12th January 2004. While the Russell 1000 presents 736 components as number of stocks during the entire period, the S&P 500 is composed by 441 assets and the Nasdaq 100 has 84 components.

5.1 Tracking Error Quantile Regression. A Dispersion Measure for the Benchmark Tracking Problem

To evaluate the goodness of the quantile regression dispersion measure of the tracking error, we empirically test it comparing some statistics in the in sample and out of sample analysis. In particular, we firstly solve index tracking portfolio selection problems with the tracking error volatility, mean absolute deviation and downside mean
semideviation as dispersion measures. Then, we solve the portfolio problem 4.5 for the tracking error quantile regression setting a spectrum of possible quantiles $\tau$ in the range $[0.01, 0.05, 0.10, 0.20, \ldots, 0.90, 0.95, 0.99]$ and developing two different strategies: the static, fixing the referred quantile a priori or the rolling when we switch the quantile at each optimization step.

In the Table 5.1, we report 5 different statistics for the in sample and out of sample analyses. Generally, we notice how the in sample analysis is not related with the out of sample and it is difficult for portfolio managers take decision based on the in sample information. Identifying the best criteria of selection, it is not possible to fix the in sample quantile and obtain the best out of sample statistics. Focusing on the alpha of the portfolio we could see how in the left side of the Table 5.1 increasing the level of the quantile we obtain higher values of portfolio alpha according to the aim to build a VaR tracking portfolio or one that wants to have better performances.

![Figure 5.1: Out of Sample Portfolio Wealth of Index Tracking Rolling Strategy, Russell 1000](image-url)
Table 5.1: Index Tracking Strategy Statistical Analysis, Russell 1000

<table>
<thead>
<tr>
<th>Russell 1000</th>
<th>In Sample</th>
<th></th>
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<td>TEDMS</td>
<td>TEV</td>
<td>#</td>
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<td>1.61</td>
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<td>1.60</td>
<td>3.05</td>
<td>62</td>
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<tr>
<td>tau 0.10</td>
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<td>1.51</td>
<td>3.91</td>
<td>49</td>
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<td>1.63</td>
<td>3.52</td>
<td>59</td>
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<tr>
<td>tau 0.40</td>
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<td>1.60</td>
<td>3.65</td>
<td>57</td>
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<tr>
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<td>3.38</td>
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<td>1.64</td>
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<td>3.22</td>
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</table>

<table>
<thead>
<tr>
<th>Out of Sample</th>
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</thead>
<tbody>
<tr>
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<td>TEDMS</td>
<td>TEV</td>
</tr>
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<tr>
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<td>18.08</td>
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<td>17.87</td>
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<td>35.04</td>
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<td>50.93</td>
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<tr>
<td>-0.23</td>
<td>34.00</td>
<td>18.02</td>
<td>47.45</td>
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</tr>
<tr>
<td>0.04</td>
<td>35.39</td>
<td>17.9</td>
<td>48.84</td>
<td>60</td>
</tr>
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<td>36.24</td>
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<td>-0.16</td>
<td>34.38</td>
<td>17.75</td>
<td>49.63</td>
<td>62</td>
</tr>
</tbody>
</table>
In a rolling framework, we switch the quantile at each optimization step according with the maximum information ratio observed in the previous month. Figures 5.1 illustrates the wealth path of the common tracking error strategies and the rolling one in the out of sample analyses.

Figure 5.1 shows the wealth paths regarding the index tracking of the Russell 1000. We notice that during the first period of the strategy the replication portfolios mimic the Russell 1000 from below with a relevant difference in 2006. Then, there are an adjustment of the portfolios during the sub-prime crisis. In this case, the proposed formulation is suitable to capture period of financial instability. The financial upturn in 2009, marks two principal features in the benchmark tracking strategies.

The comparison between the tracking error quantile regression and the common dispersion measures to mimic the performance of a benchmark highlights the theoretical and empirical impact of this measure in the benchmark tracking problem. Thus, we propose two realistic models to solve the enhanced indexation benchmark tracking
problem introducing first and second order stochastic dominance constraints. We set \( tc^+ = tc^- = 15 \text{bps} \) and the turnover constraints \( \theta = 50\% \). It means the impossibility to roll more than 50% of the invested portfolio at each optimization step. We remark that for the problem (4.6) we consider a rolling time series of 130 historical observations since the problem is still demanding considering the presence of \( 130 \times 130 = 16900 \) integer random variables.

In this approach we compute the static analysis fixing the spectrum of \( \tau \) and keeping constant the selected one for the entire period. Then, we propose the rolling strategy introducing the information ratio as decisional variable and we report the results of the wealth paths. Figure 5.2 reports the wealth paths of five static strategies to solve the enhanced indexation problem with first order stochastic dominance constraints. We notice that increasing the level of the quantile value we obtain higher returns and consequent dispersion.

![Figure 5.3: Out of Sample Portfolio Wealth of Enhanced Indexation Rolling Strategy, S&P 500](image)
Analyzing the results, we deduce the importance of stochastic dominance constraints in the portfolio selection problem. The wealth paths of the enhanced index portfolio with first and second order stochastic dominance outperform the Russell 1000 on the overall period. Only during the first period when the market is constant the portfolio with second order stochastic dominance constraints could not show extra-performances. Then, during the first investment period the enhanced indexation strategies outperform the benchmark of about 20% while at the end of 2013 the extra-performances are about 30% and 50% for FSD and SSD portfolios.

5.2 Dispersion Measure for the Benchmark Tracking Portfolio Problem and Third Order Stochastic Dominance Constraints

In this section, we propose two empirical application to the benchmarking problem. Considering the Russell 1000 stock index as the selected benchmark, we firstly address with the problem to build portfolio which mimic the performances of a stock index and then we introduce first and second order stochastic dominance constraints to evaluate the enhanced strategies in a static and rolling framework. Secondly, we propose an empirical application of the stochastic investment chain when we maximize the Rachev utility function. At each optimization step, we solve (A), (B) and (C) phases of the chain and we evaluate the out of sample wealth path of the three portfolios.

We set a spectrum of 13 possible $q \in \mathbb{R}$ in the range of the join support between portfolio and benchmark and we consider the information ratio quantile regression as decisional rule to switch from the static to the rolling approach. Figure 5.4 shows the out of sample wealth path of the rolling strategies obtained minimizing the dispersion measure derived from the $L^p$ metric. The aim of portfolio manager is to obtain portfolio as much as possible closed to the benchmark. The blue line represents the rolling strategy obtained minimizing the realistic index tracking portfolio problem while the yellow and red line represents the wealth path of the enhanced indexation problem with second and first order stochastic dominance constraints.

Analyzing the behavior of the three benchmarking strategies is clear the impact of stochastic dominance constraints in the optimization problem. Firstly, the dispersion
measure derived from the $L^P$ metric results to be a useful tool to track the benchmark performances since the blue lines mimic very well the Russell 1000. Secondly, whether second order stochastic dominance constraints produces a relative weak impact in the final wealth with gain more than 10% on the overall period, the contribution of the FSD strategy is very impressive. In fact, solving the enhanced indexation problem with

![Portfolio Wealth, Index Tracking and Enhanced Indexation](image)

**Figure 5.4:** Out of Sample Portfolio Wealth of Index Tracking and Enhanced Indexation Rolling Strategy LP Metrics, Russell 1000

first order stochastic dominance constraints we obtain portfolios which outperforms the index and the previous one for the entire investment period. In this case we have a peak in the period before the sub-prime crisis with gains about 50% and a value of the final wealth of about 2.1 times the initial one.

Then, we analyze the results of empirical applications of the stochastic investment chain problem. Figure 5.5 shows the wealth path of the three portfolios $X^A$, $X^B$ and $X^C$
during the overall period. We notice the high different behavior between them. In particular, starting from the maximization of the Rachev utility with first order stochastic dominance (blue line) we notice that the wealth increase forcefully after the sub-prime and the Greece debt crises in 2008 and 2011. However, the wealth reach a final value more than 3 times the initial one. Differently, the solution of the step (B), obtained maximizing the Rachev utility with the second order stochastic constraints where the benchmark is the previous optimal portfolio, shows a wealth path relative smooth for the entire period. In fact, comparing these two strategies we notice how the red line which represent the portfolio $X^B$ is much more conservative in the risk exposition and additive shift during the overall period.

![Portfolio Wealth, Stochastic Investment Chain Rachev Utility](image)

**Figure 5.5:** Out of Sample Portfolio Wealth of Stochastic Investment Chain Maximizing Rachev Utility Function, Russell 1000

Finally, the introduction of third order stochastic dominance constraints has the opposite behavior in the out of sample portfolio wealth. The yellow line represents the entire...
path and it shows how the portfolio is strongly exposed to upward and downward shifts. Before sub-prime crisis in 2007 the value of the portfolio was more than 3 times the initial one and after loosing all the previous gains in few months the wealth restart to increase forcefully during the following financial upturn with a final figure of about 5.3.

5.3 Linear Programming Active Management Strategy. The Maximization of Performance Measures

The proposed methodologies are applied to the active management strategy solving the problem to find a portfolio composition which beat the benchmark in the last ten years. For this reason, we compute empirical applications where the benchmarks stock index is the Russell 1000. Every strategy starts on 12th January 2004. Then, at each optimization step, we compute the \( \text{STARR}_{5\%} \) and the \( \text{RACHEV}_{5\%,2\%} \) for a total number of 125 optimization since we change portfolio composition every month (20 days). Finally, we set an upper bound level of 10\% and transaction costs (30bps) are included. We solve the six portfolio selection problem proposing portfolio approaches to address with the active management. Then, we compare and empirically test the different optimization methods and the impact to introduce stochastic dominance constraints in the problem formulation.

Figures 5.6 and 5.7 illustrate the out of sample normalized wealth path during the investment period from 12th January 2004 to 31st December 2013. In Figure 5.6 we notice how every active strategy outperforms the Russell 1000. In particular, the classical maximization of the STARR produce an extra-performance of the 30\% at the end of the investment period. Weather the introduction of the second order stochastic dominance constraints has not a strong impact in the wealth path, the first order constraints produce an increment in the portfolio gains before the sub-prime crisis. In fact, during this period the portfolio wealth gains more than 70\% while after the crisis it is difficult to enhance the performance of the maximization of the STARR.

In particular, the portfolio strategies with first and second order stochastic dominance constraints have a final wealth more than 3.5 and 3.4, respectively. Neither risk aversion nor risk seeking investors that maximize their utility solving a portfolio problem which involves the maximization of the Rachev Ratio obtain higher final value than the previous
strategies. Figure 5.7 shows the normalized wealth path of the Russell 1000 (purple line), the maximization of the Rachev ratio (azure line), the strategy with the introduction of first order stochastic dominance (red line) and the maximization of the Rachev Ratio with second order stochastic dominance constraints (gold line). Also in this case every active managed portfolio outperforms the benchmark stock index for the overall period.

Weather the common maximization of the Rachev Ratio produce a portfolio with more than 3 times the initial wealth allocation, the other two strategies have better results. In particular, the maximization of the performance measure with first order stochastic dominance constraints amplifies the market jumps during the entire period. In fact, this strategy reach 1.7 times the initial wealth in 2008 before the sub-prime crisis while investors holding this portfolio composition have also a peak of about 300% of earnings in 2010 and after a period of stability during the European sovereign debt crisis the
portfolio wealth path starts a increasing rally with a final value more than 3.5 times the original invested capital. Analyzing the two strategies with stochastic dominance constraints we notice that the first order stochastic dominance portfolio dominates the other strategies for the entire period with the exception of the last investment period. Differently from the maximization of the STARR we stress the different wealth path of the three strategies since the maximization of the Rachev Ratio is not consistent with second order stochastic dominance. Finally, figure 5.8 report the maximization of the Mean Absolute Semideviation Ratio. We observe how the three strategies outperform the benchmark represented by the Russell 1000. Moreover, the red line which illustrates the wealth path of the maximization of the MAS Ratio dominates the other for the overall period with a final gain of more than 2.1 times the initial value.
Figure 5.8: Portfolio Wealth of Active Strategies Mean Absolute Semideviation Ratio, Russell 1000
Chapter 6

Summary of Results and Conclusion

In this essay, we describe the three main areas of the benchmarking problem. This kind of problem is related to the construction of an invested portfolio which compares its performance with a given index. Considering the assets that compose the stock index, we develop index tracking, enhanced indexation and active strategies. The aim of this work is to propose theoretical and methodological approaches to cover different portfolio managers goals. In particular, we grounded our analysis on the definition of linear portfolio selection models with the introduction of different stochastic dominance constraints in the decisional problem and evaluating their benefits in terms of risk reduction and increasing gains.

After a review of the literature, in Chapter 3, we develop a new measure for the index tracking based on the quantile regression which aims to mimic the performance of a benchmark in several phases of the financial cycle considering medium and big tracked portfolios. Then, we propose a realistic model introducing a penalty function with transaction costs and turnover constraints to limit the changes in portfolio active assets. In this model, we add stochastic dominance constraints to enhance the performance of invested portfolios obtaining strategies that minimize linear and asymmetric dispersion measure, as the tracking error quantile regression, and outperform the benchmark. Empirical applications dictate the benefits of this approach in a reduction of the
active assets and portfolio turnover. Moreover, stochastic dominance allows to obtain attractive portfolio returns controlling the risk.

Then, Chapter 4 presents a generalization of the functional measure in the benchmark tracking problem reviewing the Gini tail measure and the $L^p$ metric for this kind of problem. We develop a theoretical and methodological formulation to take advantages from different orders of stochastic dominance introducing investment chain to increase the portfolio wealth or to improve the risk premium. In this framework, we linearize the portfolio problem to maximize utility functions and we introduce different types of stochastic dominance. Considering three levels to improve the portfolio construction process, we create investment strategies focusing on the behavior of non satiable risk averse investor with positive skewness. Future research will focus on an extension of this concept analyzing other possible development of the stochastic investment chain through the introduction of other order of stochastic dominance and utility functions.

Finally in Chapter 5, we address with problem to actively manage the invested portfolio to outperform the benchmark. Facing high dimensionality problem and evaluating the impact of the introduction of stochastic dominance constraints, we develop linear portfolio selection models that maximize some performance measures which are consistent with different investor’s profiles. Empirical applications show the out of sample wealth paths of these strategies and they highlight the importance of stochastic dominance in the decisional problem to obtain portfolios with a strong behavior capable to strongly produce consistent and permanent gains with respect to the benchmark.
Chapter 7

List of References


Chapter 8

List of Author’s Publications and Research


Chapter 9

Summary

This work debates several approaches to solve the benchmark tracking problems and introduces different orders of stochastic dominance constraints in the decisional process. We propose different solutions for index tracking, enhanced indexation and active managing strategies. Firstly, we introduce a linear measure to deal with the passive strategy problem analyzing its impact in the index tracking formulation. This measure results to be not only theoretically suitable but also it empirically improves the solution the results. Then, proposing realistic enhanced indexation strategies, we show how to solve this problem minimizing a linear dispersion measure.

Secondly, we generalize the idea to consider a functional in the tracking error problem considering the class of dilation, expected bounded risk measures and $L^p$ compound metric. We formulate different metrics for the benchmark tracking problem and we introduce linear formulation constraints to construct portfolio which maximizes the preference of non-satiably risk averse investors with positive skewness developing the concept of stochastic investment chain.

Thirdly, active strategies are proposed to maximize the performances of portfolio managers according with different investor’s preferences. Thus, we introduce linear programming portfolio selection models maximizing four performance measures and evaluate the impact of the stochastic dominance constraints in the ex-post final wealth.

**Keywords:** Benchmark tracking problem, dispersion measure of tracking error, performance measure, linear programming, stochastic dominance constraints.