Handicap Labelings of 4-Regular Graphs

Petr KOVAR[1,2], *Michal KRAVCENKO*[1,2], *Matej KRBECEK*[1], *Adam SILBER*[1,2]

[1]Department of Applied Mathematics, Faculty of Electrical Engineering and Computer Science, VSB–Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic
[2]IT4Innovations National Supercomputing Center, VSB–Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic

petr.kovar@vsb.cz, michal.kravcenko@vsb.cz, matej.krbecek@vsb.cz, adam.silber@vsb.cz

DOI: 10.15598/aeee.v15i2.2263

Abstract. *Let G be a simple graph, let $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijective mapping. The weight of $v \in V(G)$ is the sum of labels of all vertices adjacent to v. We say that f is a distance magic labeling of G if the weight of every vertex is the same constant k and we say that f is a handicap magic labeling of G if the weight of every vertex v is $\ell + f(v)$ for some constant ℓ. Graphs that allow such labelings are called distance magic or handicap, respectively. Distance magic and handicap labelings of regular graphs are used for scheduling incomplete tournaments. While distance magic labelings correspond to so called equalized tournaments, handicap labelings can be used to schedule incomplete tournaments that are more challenging to stronger teams or players, hence they increase competition and yield attractive schemes in which every game counts. We summarize known results on distance magic and handicap labelings and construct a new infinite class of 4-regular handicap graphs.*

Keywords

Handicap labeling, regular graph, scheduling, tournament.

1. Motivation

The motivation for handicap graphs comes from scheduling handicap incomplete tournaments which are a natural extension of an earlier problem of scheduling fair incomplete tournaments introduced by Froncek, Kovar, and Kovarova [6].

A complete tournament of n teams is represented by a complete graph in which each team plays against all $n - 1$ other teams. Such setting is usually considered fair. However, for large number of teams a complete tournament is time demanding due to numerous matches, which have to be played. Suppose ranking of teams is known, say based on the last year's performance. Obviously, the strongest team is ranked number 1, the weaker the team, the higher the rank number.

A *fair incomplete tournament* arises from a complete tournament by omitting certain matches so that every team plays the same number of matches and the total difficulty of opponents mimics the difficulty of the complete tournament for each team. On the other hand, it is easy to observe that the omitted matches also form a kind of a tournament, an *equalized incomplete tournament* in which every team plays the same number of matches and the total difficulty of opponents is the same for each team. Thus, an equalized incomplete tournament is the complement of a fair incomplete tournament and vice versa. Finally, a *handicap incomplete tournament* arises from a complete tournament by omitting matches so that every team plays the same number of matches and a certain advantage is given to weaker teams.

1.1. Distance Magic Graphs

An *equalized incomplete tournament* can be represented by a distance magic graph [6]. The vertices of the graph represent teams, while the edges represent matches played in the tournament. In this paper we identify vertices with their labels, thus by x we understand the vertex labeled x. This means that for every x we have $f(x) = x$, which simplifies our notation.

A graph G of order n with vertex set $V(G)$ and edge set $E(G)$ is called *distance magic* (see [18] or [11]) if there exists a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ such that the *weight* $w(x) = \sum_{y \in N(x)} f(y)$ of every vertex x is equal to the same magic constant k, where $N(x)$

is the set of all vertices adjacent to x. Thus, for every vertex x is $w(x) = k$. Bijection f is a distance magic labeling of G. An example of a distance magic graph is shown in Fig. 1.

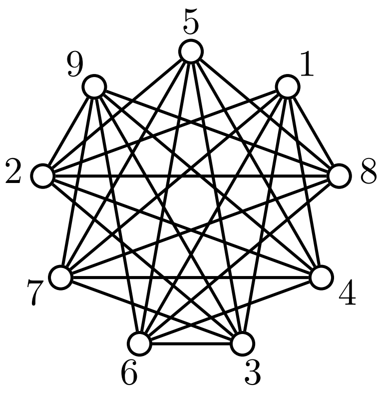

Fig. 1: A distance magic graph with magic constant $k = 30$.

1.2. Handicap Graphs

While a regular distance magic graph represents a schedule of an *equalized incomplete tournament* in which all teams have an equally strong set of opponents, a handicap graph represents a schedule of a *handicap incomplete tournament* in which certain advantage or disadvantage is given to the teams according to their strength. The weaker the team, the bigger the advantage. This would hopefully increase the attractivity of the tournament.

Formally, a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ is a *handicap labeling* of $G = (V, E)$ if there exists an integer ℓ such that the *weight* of every vertex x is $w(x) = \sum_{y \in N(x)} f(y) = \ell + f(x)$. Any graph which admits a handicap labeling is a *handicap graph* [3]. An example of a handicap graph is in Fig. 2.

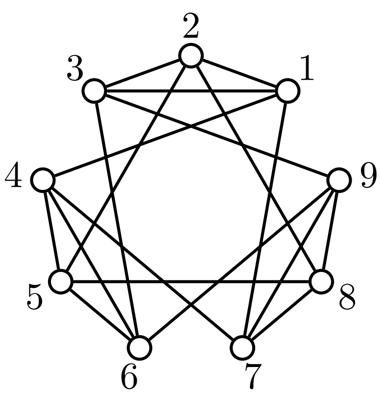

Fig. 2: A handicap graph with vertex weight $15 + f(x)$.

An extensive overview of results on magic labelings is in the dynamic survey by Gallian [8]. In this paper we show that there exist 4-regular handicap graphs of order n, such that $n \equiv 9 \pmod{18}$.

2. Known Results

2.1. Magic Rectangles

A magic rectangle is an $a \times b$ matrix with entries $1, 2, \ldots, ab$ where all row sums are equal and all column sums are equal. The existence of magic squares and magic rectangles is considered folklore and was established in several previous works. For general constructions of magic squares see e.g. [1]. Harmuth [9] and [10] gave a necessary and sufficient condition for the existence of magic rectangles:

Lemma 1. *There exists a nontrivial $a \times b$ magic rectangle if and only if $a, b > 1$, and a and b are either both odd or both even, but not both 2.*

For our purpose the existence of a $3 \times b$ matrix M with entries $1, 2, \ldots, 3b$ with the sum $3(3b + 1)/2$ in every column will suffice. Let us denote $b = 2s + 1$ for some non-negative integer s. For even b the existence is excluded by parity conditions. One possible scheme for such matrix M is given by Eq. (1). Such schemes are based on a result by Kotzig [16] and were used by Froncek [4]. Notice that the sum of every column of M is $4b + s + 2 = 4b + (b - 1)/2 + 2 = 3(3b + 1)/2$. Any $3 \times b$ magic rectangle would suffice, while the case $b = 1$ ($s = 0$) is covered by taking the column matrix with entries $1, 2, 3$.

2.2. Handicap Labelings and Related Results

Miller, Rodger, and Simanjuntak [18] completely solved the existence of distance magic labelings for some basic classes of graphs: paths, cycles, complete graphs, and trees. An easy counting argument shows that no odd-regular distance magic graph exists.

A survey paper on distance magic graphs is due to Arumugam, Froncek, and Kamatchi [2]. Moreover, the authors characterized distance magic graphs with small magic constants. All 2-regular distance magic graphs are characterized e.g. in [11], they are isomorphic to t copies of C_4. The spectrum of pairs (n, r), where n is the order of a distance magic graph and r its regularity, was completely settled for even r by Froncek, Kovar, and Kovarova in [6].

Further papers investigate distance magic graphs of odd order. It is conjectured that an r-regular distance magic graph of odd order n, $n \geq 17$ exists for all even r, such that $4 \leq r \leq n-5$. Kovar, Kovarova, and Froncek [12] proved that a 4-regular distance magic graph of an odd order n exists if and only if n is at least 17. The existence of 6-, 8-, 10-, and 12-regular distance magic graphs of odd order was completely solved by

$$M = \begin{pmatrix} 1 & 2 & \cdots & s+1 & s+2 & s+3 & \cdots & b \\ 2b & 2b-2 & \cdots & b+1 & 2b-1 & 2b-3 & \cdots & b+2 \\ 2b+s+1 & 2b+s+2 & \cdots & 3b & 2b+1 & 2b+2 & \cdots & 2b+s \end{pmatrix}. \quad (1)$$

Kovar, Silber, Kabelikova, and Kravcenko [15]. Krbecek [17] and Zacek [20] constructed 14-regular distance magic graphs for odd $n \geq 19$. For dense graphs Kovar and Silber [14] proved that an $(n-3)$-regular distance magic graph with n vertices exists if and only if $n \equiv 3 \pmod 6$ and that its structure is determined uniquely. The existence of $(n-5)$-, $(n-7)$-, and $(n-9)$-regular distance magic graphs was completely solved by Kovar and Zidek [21].

There are no r-regular handicap graphs with n vertices if both r and n are even or if $r \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, see e.g. [3], [19] and [13]. No nontrivial r-regular handicap graph with n vertices exists if $r = 1$, $r = 2$, $r = n-1$, $r = n-3$, or $r = n-2t$, such that $t \in [1, \lfloor n/2 \rfloor]$ [19] and [13].

By combining results of several authors it was possible to characterize all orders for which an odd-regular handicap graph exists. Froncek, Kovar, Kovarova, Krajc, Kravcenko, Shepanik, and Silber [7] presented r-regular handicap graphs of even order n when $n \geq 8$ and:

- $n \equiv 0 \pmod 4$ if and only if $3 \leq r \leq n-5$ and r is odd,
- $n \equiv 2 \pmod 4$ if and only if $3 \leq r \leq n-7$ and $r \equiv 3 \pmod 4$,

except when $r = 3$ and $n \in \{10, 12, 14, 18, 22, 26\}$. This is an analogy of the complete characterization of all orders for which an even-regular distance magic graph exists [6].

For even-regular handicap graphs a complete characterization of all orders and regularities is far from complete. Froncek [5] showed that an even-regular handicap graph of order n exists if and only if $n \geq 9$, except possibly when $n \in \{11, 13, 19, 23, 29\}$. A subsequent brute force search revealed that there exists no regular handicap graph of order 11, but there exist regular handicap graphs of orders $13, 19, 23, 29$.

It is conjectured that there exists an r-regular handicap graph for all odd orders $n \geq 19$ and all even regularities r, $4 \leq r \leq n-5$. A special case of this conjecture concerns 4-regular handicap graphs of an odd order n. The conjecture is stated in the Conclusion. The main result of this paper supports the conjecture.

3. Main Result

Now we show that an infinite class of 4-regular handicap graphs exists.

Theorem 1 (Main result). *Let $n \equiv 9 \pmod{18}$. Then there exists a 4-regular handicap graph with n vertices.*

Proof. The proof is constructive. Consider the graph G in Fig. 3. For $n = 18s + 9$, where s is a non-negative integer, we take $b = 2s + 1$ copies of G and a $3 \times (2s + 1)$ matrix M based on the scheme Eq. (1). Clearly, $n = 9b$.

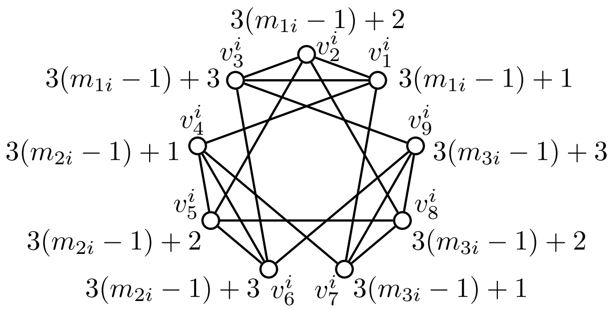

Fig. 3: Labeling of a 4-regular graph G with 9 vertices.

We label the nine vertices in the i-th copy of G for $i = 1, 2, \ldots, 2s+1$ using the entries of M, where m_{ji} is the entry in row j and column i as shown in Fig. 3. If $v_1^i, v_2^i, \ldots, v_9^i$ are the vertices of the i-th copy of G, then

$$\begin{aligned} f(v_1^i) &= 3(m_{1i}-1)+1, \\ f(v_2^i) &= 3(m_{1i}-1)+2, \\ f(v_3^i) &= 3(m_{1i}-1)+3, \\ f(v_4^i) &= 3(m_{2i}-1)+1, \\ f(v_5^i) &= 3(m_{2i}-1)+2, \\ f(v_6^i) &= 3(m_{2i}-1)+3, \\ f(v_7^i) &= 3(m_{3i}-1)+1, \\ f(v_8^i) &= 3(m_{3i}-1)+2, \\ f(v_9^i) &= 3(m_{3i}-1)+3. \end{aligned} \quad (2)$$

It remains to show that labeling f of G is handicap.

First we observe that since the entries of M are all integers from 1 to $6s+3$, the labeling f is a bijection from $V(G)$ to $[1, n] = [1, 18s+9]$. Then, the weight of vertex v_1^i is

$$\begin{aligned} w(v_1^i) &= f(v_2^i) + f(v_3^i) + f(v_4^i) + f(v_7^i) \\ &= 3(m_{1i} + m_{2i} + m_{3i}) + 3m_{1i} - 5. \end{aligned} \quad (3)$$

The sum of the i-th column of M is $m_{1i}+m_{2i}+m_{3i} = 3(3b+1)/2$, thus the weight is

$$\begin{aligned} w(v_1^i) &= 9(3b+1)/2 - 3 + 3(m_{1i}-1) + 1, \\ &= 3(9b+1)/2 + f(v_1^i), \\ &= 3(n+1)/2 + f(v_1^i). \end{aligned} \tag{4}$$

Similarly, $w(v_2^i) = f(v_1^i) + f(v_3^i) + f(v_5^i) + f(v_8^i) = 3(n+1)/2 + f(v_2^i)$ and $w(v_3^i) = f(v_1^i) + f(v_2^i) + f(v_6^i) + f(v_9^i) = 3(n+1)/2 + f(v_3^i)$. An analogous computation for the remaining vertices yields

$$w(v_j^i) = 3(n+1)/2 + f(v_j^i), \tag{5}$$

for $j = 1, 2 \ldots, 9$. Hence, comparing Eq. (5) with the definition of handicap labeling, it immediately follows that labeling f is a handicap labeling of $(2s+1)$ copies of G. This completes the proof. □

4. Conclusion

A brute force search reveals that there exists a single 4-regular handicap graph with n vertices for $n < 19$, namely graph G in Fig. 3. On the other hand we have found 4-regular handicap graphs for odd values $n \in \{19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41\}$. It seems that for less than 19 vertices the graph is simply too small to allow a distribution of labels that correspond to a handicap labeling. No 4-regular handicap graph with an even number of vertices exists [3]. We conclude this contribution by the following conjecture.

Conjecture 1. *There exists a 4-regular handicap graph with n vertices if and only if $n = 9$ or $n \geq 19$ and n is odd.*

Theorem 1 justifies the conjecture for all integer values $n \equiv 9 \pmod{18}$.

Acknowledgment

This work is supported by The Ministry of Education, Youth and Sports from the National Programme of Sustainability (NPU II) project "IT4Innovations excellence in science – LQ1602". The work of the second, third, and fourth author is partially supported by Grant of SGS No. SP2017/122, VSB–Technical University of Ostrava, Czech Republic. The work of the fourth author is partially supported by grant of SGS No. SP2017/182 "Solving graph problems on spatio-temporal graphs with uncertainty using HPC", VSB–Technical University of Ostrava, Czech Republic.

Finally, we wish to express our thanks to the referees for their suggestions, which helped to improve the readability of the text significantly.

References

[1] ANDREWS, W. S. *Magic Squares and Cubes.* New York: Dover, 1960. ISBN 0486206580.

[2] ARUMUGAN, S., D. FRONCEK and N. KAMATCHI. Distance Magic-Graphs - A Survey. *Journal of the Indonesian Mathematical Society.* 2011, vol. 17, iss. 3, pp. 11–26. ISSN 2086-8952. DOI: 10.22342/jims.0.0.15.11-26.

[3] FRONCEK, D. Handicap distance antimagic graphs and incomplete tournaments. *AKCE International Journal of Graphs and Combinatorics.* 2013, vol. 10, iss. 2, pp. 119–127. ISSN 0972-8600.

[4] FRONCEK, D. Fair incomplete tournaments with odd number of teams and large number of games. *Congressus Numerantium.* 2007, vol. 187, iss. 1, pp. 83–89. ISSN 0316-1282.

[5] FRONCEK, D. Regular handicap graphs of odd orders. *Journal of Combinatorial Mathematics and Combinatorial Computing.* In Press.

[6] FRONCEK, D., P. KOVAR and T. KOVAROVA. Fair incomplete tournaments. *Bulletin of the ICA.* 2006, vol. 48, iss. 1, pp. 31–33. ISSN 1183-1278.

[7] FRONCEK, D., P. KOVAR, T. KOVAROVA, B. KRAJC, M. KRAVCENKO, M. SHEPANIK and A. SILBER. On regular handicap graphs of even order. *Electronic Notes in Discrete Mathematics.* In Press.

[8] GALLIAN, J. A. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics.* 2016, vol. 19, iss. DS6, pp. 1–408. ISSN 1077-8926.

[9] HARMUTH, T. Ueber magische Quadrate and aehnliche Zahlenfiguren. *Archiv der Mathematik und Physik, Vol. 66.* C. A. Koch, 1881.

[10] HARMUTH, T. Ueber magische rechtecke mit ungeraden Seitenzahlen. *Archiv der Mathematik und Physik, Vol. 66.* C. A. Koch, 1881.

[11] JINNAH, M. I. On Σ-labelled graphs. In: *Technical Proceedings of Group Discussion on Graph Labeling Problems.* 1999, pp. 71–77.

[12] KOVAR, P., T. KOVAROVA and D. FRONCEK. A note on 4-regular distance magic graphs. *Australasian Journal of Combinatorics.* 2012, vol. 54, iss. 1, pp. 127–132. ISSN 1034-4942.

[13] KOVAR, P., T. KOVAROVA, B. KRAJC, M. KRAVCENKO and A. SILBER. On regular handicap graphs. Manuscript.

[14] KOVAR, P. and A. SILBER. Distance magic graphs of high regularity. *AKCE International Journal of Graphs and Combinatorics*. 2012, vol. 9, iss. 2, pp. 213–219. ISSN 0972-8600.

[15] KOVAR, P., A. SILBER, P. KABELIKOVA and M. KRAVCENKO. On Regular distance magic graphs of odd order. Submitted.

[16] KOTZIG A. On magic valuations of trichromatic graphs. In: *Reports of the CRM (CRM-148)*. University of Montreal, 1971.

[17] KRBECEK, M. *Applications of 1-VMV labelings graphs*. Ostrava, 2012. Bachelor thesis. VSB–Technical University of Ostrava. Supervisor P. Kovar.

[18] MILLER, M., C. RODGER and R. SIMANJUNTAK. Distance magic labelings of graphs. *Australasian Journal of Combinatorics*. 2003, vol. 28, iss. 1, pp. 305–315. ISSN 1034-4942.

[19] SHEPANIK, A. *Graph Labelings and Tournament Scheduling*. Minnesota, 2015. Master thesis. University of Minnesota Duluth. Supervisor D. Froncek.

[20] ZACEK, J. *Utilization of 1-VMV labelings of graphs*. Ostrava, 2012. Bachelor thesis. VSB–Technical University of Ostrava. Supervisor P. Kovar.

[21] ZIDEK, A. *Incomplete sport tournaments*. Frydlant nad Ostravici, 2012. Research project. Supervisor P. Kovar.

About Authors

Petr KOVAR was born in Brno, Czech Republic. He received his Mgr. from Masaryk University in Brno in 1998, his M.Sc. from University of Minnesota Duluth in 2004, and his Ph.D. from VSB–Technical University of Ostrava in 2004. His research interests include discrete mathematics, graph theory and their applications.

Michal KRAVCENKO was born in Moscow, Russia. He received his M.Sc. from VSB–Technical University of Ostrava in 2013. His research interests include discrete mathematics, numerical analysis, computer science, graph theory and their applications.

Matej KRBECEK was born in Olomouc, Czech Republic. He received his M.Sc. from VSB–Technical University of Ostrava in 2014. His research interests include discrete mathematics, graph theory and their applications.

Adam SILBER was born in Ostrava, Czech Republic. He received his M.Sc. from VSB–Technical University of Ostrava in 2013. His research interests include discrete mathematics, graph theory and their applications.