Summary

The paper represents an introductory study about a superconductive cable levitating in Earth’s magnetic field. Built are two mathematical models of the problem providing both the shape of the arc of the cable and forces acting along it. The theoretical analysis is supplemented with an illustrative example.

1. INTRODUCTION

Earth’s magnetic field is produced by rotation of liquid and electrically conductive shell 3 of Earth’s core 2 with respect to relatively unmovable shell 1, see Fig. 1a (another view is in Fig. 1b).

![Fig. 1a: Earth magnetic field](image)

This field whose strength on the northern hemisphere reaches values between 38–56 A/m (which depends on the structure of the corresponding lithospheric plates) represents one of very important attributes of the planet Earth. The human population employs this field

- unconsciously – the field represents an “umbrella” protecting Earth’s surface against the impact of electrically charged particles arriving to Earth in the consequence of the solar activity and also from space and
- consciously – Earth’s magnetic field is used for
  - orientation on Earth’s surface – the first magnetic compasses were built in China, about 600 years before Christ,
  - geological survey – subsurface deposits of ferromagnetic ores produce local anomalies in Earth’s magnetic field,
  - determination of the time scales in geology and also in archeology.

Nowadays, however, we can often see various considerations aimed at its nontraditional employment. Mentioned can be, for example, “tethers” (particulars can be found in [1] and [2]), which are superconductive cables unwound in a suitable direction from various space objects (sattelites, the last sections of booster rockets, space labs etc.) that can be used as:

- Electric voltage source – a suitably oriented conductor of length l moving at a velocity \( v \) in magnetic field of flux density \( B \) induces voltage \( u_i = \int_0^l (v \times B) \cdot dl \) [3]. Transport of such a cable to the orbit is much cheaper than transport of classical photovoltaic cells.
- A suitably oriented conductor of length l carrying current I and moving in magnetic flux density \( B \) is affected by the Lorentz force of value \( F_l = I \int_0^l (dl \times B) \) [3] that deccelerates or accelerates the space object and shifts it to a lower or higher orbit.

These nontraditional space applications that are intensively studied and now already also experimentally validated (see, for example, [1] – projects TSS-1a Oidipus-C, NASA+Italian Space Agency) could also be transferred to Earth’s surface. The paper deals with one possible application – possibility of employment of Earth’s magnetic field for levitation of a superconductive cable closely above Earth’s surface, which could be used, for instance, in meteorology, where the levitating cable could replace the trial balloons, particularly for investigation of lower layers of the atmosphere.

![Fig. 1b: Earth magnetic field with the indicated position of the cable (see Figs. 3a, b)](image)
2. FORMULATION OF THE PROBLEM

Fig. 2. depicts an arrangement of a typical superconductive cable for earthly conditions. A similar cable could play a role in the considered case.

![Fig. 2: Arrangement of the superconductive cable](image)

Its superconductive system consists of a greater amount of thin Cu tubes 2 filled in with the actual superconductor and placed in Kevlar shell 1. The shell representing the carrying element is flown through liquid He 3 that secures the superconductive regime of the cable.

The cable of starting length \( s_0 = 2l \) is located on Earth’s surface (near the equator) in the horizontal position between points P and Q (see Fig. 1b) whose abscissa is oriented perpendicularly to the force lines of Earth’s magnetic field. Near the equator we can consider Earth’s magnetic field approximately parallel to Earth’s surface, so that its flux density \( B = z_0 B \) (see Fig. 3a).

![Fig. 3a: The investigated arrangement of a superconducting cable in the domain with uniform flux density – starting position](image)

The points P and Q represent the reel drums equipped with brakes. The cable is wound on them and after their braking off it can freely unwind to a general length \( s > 2l \) (Fig. 3b). If the cable carries current \( I \) in the indicated direction, it begins to be affected by the total Lorentz force

\[
F_L = I \int_{-l}^{l} (dl \times B)
\]

oriented in direction \( y \).

This force then lifts the cable upwards to a general position \( y = f(x) \) corresponding to the “braked off” length \( s \) of the cable. In this position, however, the specific Lorentz force acting on the unit length of the cable has generally two components \( f_L = x_0 (\pm f_{Lx}) + y_0 f_{Ly} \). Thus, the levitation effect of Earth’s magnetic field can be expected to decrease with growing length \( s \) of the cable.

![Fig. 3b: The investigated arrangement – final position of the cable](image)

The aim of the paper is to evaluate the

- dependence \( y = f(x) \) on the “braked off” length \( s \) of the cable and on the value \( I \) of the superconductive current,
- dependence of the distribution of forces acting on the cable on the same parameters,
- restriction of the considered levitating effect by the “braked off” length \( s \) and current \( I \).

3. MATHEMATICAL MODELS OF THE PROBLEM AND ITS SOLUTION

The mathematical description of the problem may be carried out in two ways:

- Solution of a nonlinear ordinary differential equation formulated as an initial problem and expressing the force and torque balance in an infinitesimal element of the cable.
- Solution of a system of algebraic equations expressing only the balance of forces in a finite element of the cable (the torque balance is here satisfied automatically)

Each of these two models has its advantages and drawbacks. The differential model allows using of numerical algorithms or professionally sophisticated function procedures providing both convergence and required accuracy of solution of the corresponding equation. On the other hand, the algebraic model is more flexible and provides an easier realization of some partial, specific computations such as local distribution of external forces, suppression of deformation of parts of the superconductive cable etc. That is why we used both models that provided a good accordance of the results.
3.1. Differential mathematical model

For obtaining the differential equation we start from Fig. 4a and Fig. 4b showing the lifted cable and situation in its element.

Here $H_1, H_2$ denote the horizontal components of forces at points 1 and 2, $V_1, V_2$ the vertical components and $q$ is the unit weight of the cable.

First let us express the particular components of the Lorentz forces. From Fig. 4b we can easily derive that

$$dF_{Lx} = -Bldx \cdot \frac{y'}{\sqrt{1+y'^2}}, \quad dF_{Ly} = Blds \cdot \frac{1}{\sqrt{1+y'^2}}$$

(1)

$$dF_1 = gds \cdot B, \quad dF_2$$

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(1)

and as $ds = \sqrt{1+y'^2} \, dx$ we immediately have

$$dF_{Lx} = -Bldy, \quad dF_{Ly} = Bldx.$$  \hspace{1cm} (2)

The balance equations for the forces in the element read

$$H_1 + dF_{Lx} = H_2,$$

$$V_1 - dF_{Ly} - qds = V_2$$

(3)

and for the torque with respect to point 2

$$dF_{Ly} \frac{dx}{2} - V_1 dx + dF_{Lx} \frac{dy}{2} = qds \cdot \frac{dx}{2} - H_1 dy.$$  \hspace{1cm} (4)

As at any point of the cable $V = H'y'$, we immediately have

$$dV = d(H'y') = H \cdot dy' + y'dH$$

(5)

where (see (3))

$$dH = H_2 - H_1 = dF_{Lx},$$

$$dV = V_2 - V_1 = qds - dF_{Ly}.$$  

After substituting from (2) and for $ds$ we finally have

$$q\sqrt{1+y'^2} \, dx - Bldx = H \cdot dy' + y'Bl dy$$

and hence

$$q\sqrt{1+y'^2} - Bl(1+y'^2) = H \cdot y''.$$”

(6)

which represents a nonlinear differential equation describing the resultant shape of the cable. Force $H$ is given as (see (2))

$$H = Bl + H_0$$

(7)

where $H_0$ is the horizontal tension in the cable at its beginning ($x = l$). The first initial condition read $y(l) = y(-l) = 0$, the second one is the known length $s$ of the cable.

3.2. Algebraic mathematical model

Consider an arrangement of the superconductive cable in Cartesian coordinate system as is depicted in Fig. 3a. In direction $x$ the distance $1/2$ is divided into $N_e$ uniform elements $\Delta x$ (Fig. 5a) with an equivalent number of corresponding nonuniform elements $\Delta s_i$ on the curve $s = y(x)$ describing the shape of arc of the cable. To a general distance $x_i$ we assign the general length $s_i$ of the corresponding arc of the cable.

$$F_{ex} 0 1 \Delta x 2 i-1 i i+1 N_e x-i$$

$$F_{0} s_i \Delta s_i F_{ej} F_{ej}$$

Fig. 5a: Distribution of forces on the cable

The part $s_i$ of the cable is affected by the following forces (Fig. 5b):
• \( F_{s0} \), \( F_{s,i} \) – internal forces producing tension in the cable,
• \( f_{L,i} \) – the Lorentz forces acting perpendicularly on particular elements \( \Delta s_i \),
• \( F_{g,i} \) – weight of particular elements \( \Delta s_i \),
• \( F_{\text{ext}} \) – external force loading the cable.

According to Figs. 5a, 5b we have
\[
F_{s0} = -x_0 F_{00}, \quad F_{\text{ext}} = \Psi F_{\text{ext}}, \\
F_{s,i} = x_0 F_{s,i} + \Psi_{s} F_{s,i}, \quad F_{s0} = |F_{s0}|, \\
F_{\text{ext}} = |F_{\text{ext}}| F_{g,i} = \Psi_{g} F_{g,i}, \quad F_{g,i} = f_g \Delta s_i, \\
F_{s,i} = \left| F_{s,i} \right| \cos \alpha_i, \quad F_{g,i} = \left| F_{g,i} \right| \sin \alpha_i.
\]

At the same time there holds
\[
\Delta x = \frac{1}{N_e}, \quad \Delta s_i = \sqrt{\Delta x^2 + \Delta y_i^2}, \\
\sin \alpha_i = \frac{\Delta y_i}{\Delta s_i}, \quad \cos \alpha_i = \frac{\Delta x}{\Delta s_i}.
\]

The conditions of balance of particular forces now read
\[
x : -F_{s0} + \sum_{k=1}^{i} f_{L,k} + F_{s,i} = 0, \quad \psi : F_{\text{ext}} - \sum_{k=1}^{i} f_{L,k} + f_{g,i} = 0.
\]

After substitution from (8) and (9) to (10) we obtain
\[
-F_{s0} + B I \sum_{k=1}^{i} \Delta s_k \sin \alpha_k + F_{s,i} \cos \alpha_i = 0 \Rightarrow \\
F_{s0} = -B I \sum_{k=1}^{i} \Delta s_k \sin \alpha_k \\
F_{s,i} = \frac{F_{s0} - B I \sum_{k=1}^{i} \Delta s_k \sin \alpha_k}{\cos \alpha_i}
\]

Analogously for the components in direction \( \psi \) we have
\[
F_{\text{ext}} = -B I \sum_{k=1}^{i} \Delta s_k \cos \alpha_i + f_{g,i} \sum_{k=1}^{i} \Delta s_k + F_{s,i} \sin \alpha_i = 0 \Rightarrow \\
F_{s,i} = \left( -F_{\text{ext}} + B I \sum_{k=1}^{i} \Delta s_k \cos \alpha_i - f_g \sum_{k=1}^{i} \Delta s_k \right) / \sin \alpha_i.
\]

Comparison of (11) and (12) provides
\[
F_{s0} = -B I \sum_{k=1}^{i} \Delta s_k \sin \alpha_k \\
\quad \cos \alpha_i
\]

The algorithm of solution of this mathematical model may be realized in the following steps:
• For some value of \( F_{s0} \) solution of (13) by, for example, the Regula Falsi method using (9), provides, by means of (4), the value of \( \psi_i \).
• Solution of (11) with respect to (8) provides for the value \( \psi_i \) the force \( F_{s,i} \).
• The first two steps are repeated for all values of \( i = 1, K, N_e \); at the same time we calculate the value of \( \sum_{k=1}^{N_e} \Delta s_k \) that is necessary for correction of \( F_{s0} \).
• Condition \( \sum_{k=1}^{N_e} \Delta s_k = s \) then provides (again using the Regula Falsi method) the value of \( F_{s0} \).

4. COMPUTER MODEL, ACHIEVED ACCURACY OF SOLUTION

Solution of the mathematical model 3.1 (equation (6)) was carried out by the fourth-order Runge-Kutta method written in Matlab. Proved was very good convergence of the numerical process – for finding of \( y(x) \) in the interval \( 0 \leq x \leq 50 \text{ m} \) with accuracy of 3 valid digits we needed only 50 steps.

Solution of the mathematical model 3.2 was performed by a user program written in Borland Delphi. The convergence of the numerical process for finding both \( y(x) \) and \( F_s(x) \) was good again. Accuracy of 3 valid digits was reached in about 200 steps.

5. ILLUSTRATIVE EXAMPLE

5.1. Technical specification

Considered is an arrangement of the superconducting levitating cable according to Fig. 2. Its dimensions and all physical parameters are listed in Tab. 1. It is necessary to carry out such a set of testing computations that would provide answers to questions in paragraph 2.

5.2. Results and their discussion

Dependencies \( s = y(x) \) on the braked-off length of the cable and current \( I \) follow from Fig. 6.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Unit</th>
<th>Value</th>
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Fig. 5b: Forces in an element of the cable
Levitation of superconductive cable in earth magnetic field

We can see that the lift increment of the cable decreases with the length \( s \) of the braked-off cable. It is obviously caused by the change of orientation of vector \( f_L(s) \) along the length \( s \) when this length changes, see Figs. 8 and 9. Increase of lift \( y(x) \) could be, of course, achieved by diminishment of specific mass \( f_s \) of the cable, by growth of current \( I \), and, especially, by growth of distance \( PQ = 2l \).

Evaluation of distribution of the forces acting on the cable on its braked-off length \( s \) and current \( I \) can be carried out from Figs. 7, 8 and 9.

Fig. 7: Dependence of the internal force \( F_s \) on length \( s \) 
\[(I = 1.5\cdot10^6 \text{A})\]

Fig. 8: Dependence of component \( f_{L,x} \) of the Lorentz force \( f_L \) on length \( s \) \((I = 1.5\cdot10^6 \text{A})\)

Fig. 9: Dependence of component \( f_{L,y} \) of the Lorentz force \( f_L \) on length \( s \) \((I = 1.5\cdot10^6 \text{A})\)

Fig. 6: Dependence of the lift \( y \) of the cable on length \( s \) 
\[(I = 1.5\cdot10^6 \text{A})\]
The paper shows that the idea of a levitating superconductive cable in Earth’s magnetic field is, from the viewpoint of the theoretical principles, quite real. Its practical realization would require, however, solution of a number of technological problems (a sufficiently firm and light superconductive cable, easy generation of high current, transport of liquid He to the unwound cable etc.) and also problems of economic character.

Further research of theoretical questions in the domain should be aimed at:
- possibilities of increase of the perpendicular component $F_L$ of the Lorentz force by increase of stiffness of a certain part of the cable,
- solution of the task as a fully 3D problem (respecting of effects of wind etc.).

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REFERENCES