Robust optimization with nonnegative decision variables: A DEA approach

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ABSTRACT

Robust optimization has become the state-of-the-art approach for solving linear optimization problems with uncertain data. Though relatively young, the robust approach has proven to be essential in many real-world applications. Under this approach, robust counterparts to prescribed uncertainty sets are constructed for general solutions to corresponding uncertain linear programming problems. It is remarkable that in most practical applications, the variables represent physical quantities and must be nonnegative. In this paper, we propose alternative robust counterparts with nonnegative decision variables – a reduced robust approach which attempts to minimize model complexity. The new framework is extended to the robust Data Envelopment Analysis (DEA) with the aim of reducing the computational burden. In the DEA methodology, first we deal with the equality in the normalization constraint and then a robust DEA based on the reduced robust counterpart is proposed. The proposed model is examined with numerical data from 250 European banks operating across the globe. The results indicate that the proposed approach (i) reduces almost 50% of the computational burden required to solve DEA problems with nonnegative decision variables; (ii) retains only essential (non-redundant) constraints and decision variables without alerting the optimal value.

1. Introduction

The robust optimization has been proposed to handle uncertainties in the input data in classical mathematical programming problems. An alternative approach that immunizes uncertain parameters in some probability sense is that of the stochastic programming which dates back to Dantzig (1955), See Prékopa (1995), Birge & Louveaux (1997) for references. Although the robust approach was introduced by Soyster (1973), it was until the late 1990s that it took a massive flurry of interest in the mathematical programming community. Since then, several robust models have been proposed (see Ben-Tal & Nemirovski, 1998, 1999, 2000; Bertsimas & Sim, 2004; Bertsimas, Pachamanova, & Sim, 2004; Ben-Tal, El Ghaoui, Nemirovski, Arkadii, & Nemirovski, 2009) and the field continues to be explored due to its usefulness in application. The standard robust optimization adopts a conservative methodology that confines all uncertain parameters to a pre-defined uncertainty set so as to optimize the worst-case performance for all feasible realization of the uncertain parameters in the defined set. It is conceivable that, like the stochastic programming, the robust optimization approach leaves some theoretical and practical issues to be addressed. Key among these issues include (i) structure of the uncertainty set, (ii) tractability of the robust formulation, (iii) conservativeness and probability guarantees to the distribution of the uncertain parameters in the uncertainty set, (iv) complexity of the robust models and (v) quality of the robust solution (Bertsimas, Brown, & Caramanis, 2011; Gorissen, Yannakakis, & den Hertog, 2015). In consequence, rather than finding a usual optimal solution, the concern has been to seek the best performance under most realizations of the uncertain parameters.

Most remarkably, given the tractability of most robust linear programming, the robust technique has sparked interest in many applications in management science. Diverse areas of operations research applications including portfolio optimization, statistics, and learning, supply chain and inventory management, engineering etc. have been considered in the literature. For a comprehensive survey, we refer the reader to Bertsimas et al. (2011). However, despite the empirical success in these areas, the robust optimization has come under some practical concerns. As mentioned earlier, one of the critical practical issues of concern is the computational cost relating to the robust models. Although this issue is usually addressed with a flexible selection of uncertainty set, notwithstanding, it is important to note that the robust approaches considered usually provide general solutions of which some could be negative. In other words, the robust counterpart models provided in literature are generally defined with free-in-sign decision variables and a separate study looking at only nonnegative

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decision variables is not available. It is remarkable that in most practical problems, the variables represent physical quantities which are nonnegative (Bazaraa, Jarvis, & Sherali, 2010). Therefore, where decision variables are only nonnegative, a significant computational disadvantage is that the general robust counterpart models proposed in the literature present “unwanted parameters” that demand more computational resources. As a result, we believe that robust counterparts with nonnegative decision variables are needed. That is, there is the need for robust formulations, which by virtue of rendering some parameters redundant in the classical robust optimization models yield solutions faster than the former. In this paper, we consider different robust counterpart models and formulate alternative models when decision variables are nonnegative. We call them reduced robust counterparts (RRC). These reduced models are equivalent to the former models without any redundant variable or constraint. As a result, the problem size (that is the number of variables and constraints) of reduced model is significantly decreased which point out that the later models are more concise, more reliable, and more application-driven than the former models. In our pursuit of this cause, we use data envelopment analysis (DEA) method as an application-driven example to illustrate the practicality and computational usefulness of the RRC models. The DEA is one of the well-known linear programming (LP) applications that most often involves nonnegative variables.

DEA is an LP approach for measuring the relative performance of homogenous decision-making units (DMUs) with multiple inputs and multiple outputs. The DEA is a non-parametric and data-oriented methodology which does not require any specification of the functional form of the input-output relationship as compared to parametric approaches, like stochastic frontier analysis (SFA), that require prescribing the functional forms. The DEA model was initially proposed by Charnes, Cooper, & Rhodes (1978) as a mathematical programming technique to evaluate DMUs under constant returns-to-scale (CRS) assumption. Subsequent developments including the variable returns-to-scale (VRS) models (e.g. Banker, Charnes, & Cooper, 1984), robust efficiency analysis (e.g. Sadjadi, Omrani, Abdollahzadeh, Alinaghian, & Mohammadi, 2011; Omrani, 2013), and robust Russell and enhanced Russell measures (Salahi, Toloo, & Hasabirad, 2018) have shown the DEA to be an essential tool for performance measurement in banks, education, healthcare, energy etc. The DEA drives weight directly from data and provides a positive efficiency score (less than or equal to 1) measured as the weighted sum of outputs to the weighted sum of inputs. Under this paradigm, DMUs are classified into two mutually exclusive and collectively exhaustive groups; efficient (when the efficiency score is 1) and inefficient (when the efficiency score is less than 1).

Recent research in DEA has focused on the robust optimization application in DEA to ensure robustness in efficiency analysis. Although the field is quite new and developing, variants robust models encompassing different uncertainty sets and scenarios have been explored and introduced into the DEA (see Sadjadi & Omrani, 2008; Atci & Gülpmar, 2016). It is important to note that, equality constraints containing uncertain parameters restrict the feasible region and may lead to infeasibility issue (for more details see Ben-Tal et al., 2009, Chapter 2). The multiplier form of DEA models involves a normalization constraint which is in an equation form containing uncertain input data. Accordingly, most researchers find it difficult handling uncertainties in the inputs and outputs data simultaneously. Regarding the normalization constraint, an alternative formulation in which the constraint is feasible for all data uncertainties is adopted in this paper for a feasible robust DEA. In other words, using a proposed formulation, we provide a coherent feasibility treatment to the normalization constraint as with all robust optimization compared to the treatment from other studies (c.f. Omrani, 2013; Salahi, Torabi, & Amir, 2016) and therefore may be regarded a novelty approach for robust DEA to general uncertainty modeling. Moreover, unlike in previous studies, the robust DEA considered for computational studies is based on the aforementioned RRC. In summary, the main contributions of the paper are the following:

1. We propose new robust counterpart optimization models with nonnegative decision variables. This leads to an approach which is more applicable and computationally cost-effective for problems involving nonnegative decision variables.
2. The suggested robust approach is used to propose a new robust DEA model. Our robust DEA models (called reduced robust DEA) are compared to existing robust DEA models.
3. Prior to the reduced robust DEA formulation, we adjust the equality constraint in the normalization of the multiplier DEA models to inequality in order to allow for feasible and simultaneous consideration of uncertainties in the inputs and outputs data.
4. We consider a case study of 250 banks in Europe to validate our new approach. The obtained results point out that the proposed robust DEA model reduces 50% of the required computational burden.

The structure of the paper is organized as follows: In Section 2, we review different robust counterparts models to the most commonly used uncertainty sets. Section 3 presents the robust counterparts models with nonnegative decision variables. In addition, the theoretical complexity of these models is analyzed which indicates less iteration required to solve the reduced models by any efficient algorithm. The DEA and robust DEA approaches are presented in Sections 4 and 5 which includes dealing with equality constraint in uncertain DEA. The proposed approach: reduced robust DEA is presented in Section 6. We provide a practical banking problem and test the complexity of the models in Section 7. The paper ends with conclusion and remarks in the next section.

2. General robust counterpart formulations

In this section, we review different uncertainty sets and demonstrate their robust counterpart formulations. To this end, we first consider an uncertain linear programming model

\[
\begin{align*}
\zeta = & \max \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \sum_{j=1}^{m} \bar{a}_{ij} x_j \leq b_i \quad i = 1, \ldots, m \\
& l_j \leq x_j \leq u_j \quad j = 1, \ldots, n
\end{align*}
\]

where \((c_1, \ldots, c_n)\) is a cost coefficient vector, \(\bar{a}_{ij}\) represents the value of the technological coefficient that is subject to uncertainty, \((b_1, \ldots, b_m)\) is the right-hand-side vector, \((x_1, \ldots, x_n)\) is decision variable vector, and \(l_j, u_j\) are the lower and upper bounds for decision variable \(x_j\). Let \(X\) be the feasible region of model (1), i.e.,

\[X = \{ x : \bar{a} x \leq b, \forall i, l \leq x \leq u \} \subset \mathbb{R}^n \]

where \(\bar{a} = (\bar{a}_1, \ldots, \bar{a}_m)\), \(l = (l_1, \ldots, l_n)\), and \(u = (u_1, \ldots, u_n)\). By standard transformations, we can assume without loss of generality that \(l_j\) is finite for each \(j = 1, \ldots, n\). In addition, assume that only the technological coefficients are subjected to uncertainty and whils their distribution may be unknown, they are known to be symmetric in an interval. Thus, let \(l_j, u_j\) represent the set of coefficients in row \(i\) that are subject to uncertainty, then the true value of each entry \(a_{ij}\), \(j \in J_i\) is modeled as a symmetric and bounded variable taking values in the interval \([a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]\) (Bertsimas & Sim, 2004). The true value of the uncertain technological coefficient can be expressed as \(\bar{a}_{ij} = a_{ij} + \gamma_{ij} \delta_{ij}\) where \(a_{ij}\) is the nominal value, \(\bar{a}_{ij}\) is the maximum distance that specifies how much the nominal value is likely to deviate from the true value, and \(\gamma_{ij}\) denotes random variable that is symmetrically distributed in the interval \([-1, 1]\). Suppose an uncertainty set \(\mathcal{U}\) (convex in structure) is constructed as an immunization region for the uncertain technological coefficients, the general robust counterpart (GRC) to a predefined uncertainty set for the classical uncertain LP model (1) can be formulated as:
let $z_* = \max \sum_{j=1}^n c_j x_j$

\[\sum_{j=1}^n a_j x_j + \max_{\eta \in \mathcal{E}} \left\{ \eta_j \bar{a}_j x_j \right\} \leq b_i, \quad i = 1, \ldots, m\]

\[l_j \leq x_j \leq u_j, \quad j = 1, \ldots, n\]

Let $X_{GRC}$ be the feasible region of model (2) involving the inner maximization problem

\[\max_{\eta \in \mathcal{E}} \left\{ \sum_{j=1}^m \eta_j \bar{a}_j x_j \right\} \quad \text{i.e.,} \quad X_{GRC} = \{ x : a x + \max_{\eta \in \mathcal{E}} \left\{ \sum_{j=1}^m \eta_j \bar{a}_j x_j \right\} \leq b_i \land i, \ I \leq x \leq u \} \subseteq \mathbb{R}^n.\]

A vector $x$ is a robust feasible solution if $x \in X_{GRC}$. Note that $X_{GRC} \subseteq X$ which follows from the fact that the optimal objective value of the GRC model (2) is less than or equal to the optimal objective value of the uncertain LP model (1), i.e., $z_* \leq z$. In other words, taking uncertainty into consideration does not lead to improving the optimal objective value. Rather, any optimal solution $x^* \in X_{GRC}$ corresponds to a solution that maximizes the worst-case objective function $\sum_{j=1}^n c_j x_j$ under all realizations of $\eta_j \in \mathcal{E}$. Such worst-case solutions are obtained by first taking the dual of the inner maximization, which by the strong duality property (see Bazaraa et al., 2010) yield the same optimal objective values as its dual.

Any solution to a specific robust counterpart to model (2) depends on the structure of the uncertainty set. The tractability of the robust counterpart also depends on this set. For instance, suppose $\mathcal{E}$ is convex and the constraints are feasibly bounded in a convex region, then model (2) would lead to a computationally tractable solution (Ben-Tal et al., 2009). The general structure of the uncertainty set $\mathcal{E}$ is related to the distribution of uncertain parameters. For an unbounded distribution, the box, ellipsoidal, and polyhedral uncertainty sets can be used for the robust counterpart whereas interval constraint is necessary for bound distribution. The consideration of the latter case is necessary to avoid a more conservative solution. The box, interval + ellipsoidal, and interval + polyhedral uncertainty sets are reviewed in this paper. As we mentioned earlier, the hint to deriving the robust solution to these uncertainty sets involves solving the subproblem (inner maximization problem) in model (2) using duality construction. For detailed information on these constructions, see Yuan, Li, & Huang (2016).

2.1. Robust counterpart to interval uncertainty set

In the robust optimization framework, the random variables, $\eta$, are assumed to be independent in the interval $[\alpha_i, \beta_i]$ and the distribution over this interval is determined by the nature of the uncertain parameters. For the LP model (1), consider that the perturbation bound for the uncertain coefficients is given by $\Phi$. A box/interval uncertainty set created by the interaction of perturbation of the random variables can be described as follows:

\[\mathcal{E}_{int}(\Phi, \Omega) = \left\{ \bar{a}_j = a_j + \eta_j \bar{a}_j | \| \eta \|_{\Omega} \leq \Phi \right\}\]

The simplest case where knowledge about the distribution of $\eta$ is known and the probability guarantees are given is when $\Phi = 1$ and $E(\eta) = 0$ (see Ben-Tal et al., 2009). The parameter $\Phi$ varies between 0 and 1 and the optimization model is robust when all the uncertain coefficients can be realized within the bound provided by the uncertainty set (3). The interval uncertainty set is shown in Fig. 1a for the distribution of random variables taking values within the bound $\Phi = 1$. Note that $\| \eta \|_{\Omega} = 1$ coincides with the highly conservative robust formulation of Soyster (1973).³ Thus, taking $\mathcal{E} = \mathcal{E}_{int}(1)$ in model (2), the subproblem $\max_{\eta \in \mathcal{E}_{int}(1)} \left\{ \sum_{j=1}^m \eta_j \bar{a}_j x_j \right\}$ for $i = 1, \ldots, m$ are linear optimization problems and hence substituting their related duals leads to the following robust counterpart:

\[\max \sum_{j=1}^n c_j x_j\]

\[\sum_{j=1}^n a_j x_j + \max_{\eta \in \mathcal{E}} \left\{ \eta_j \bar{a}_j x_j \right\} \leq b_i, \quad i = 1, \ldots, m\]

\[l_j \leq x_j \leq u_j, \quad j = 1, \ldots, n\]

Note that Soyster (1973) added a nonnegative variable $y_j$ for each variable $x_j$; however, variable $y_j$ is redundant if $\| \bar{a}_j \| \leq 1$. As a result, we consider variable $y_j$ if at least one of the coefficients of $x_j$ is a constraint is uncertain. Note also that an accurate representation of the indices further reveals the actual number of variables and constraints in the model including those that are uncertain.

2.2. Robust counterpart to combined interval and ellipsoidal uncertainty set

Following the over-conservativeness of robust solutions using the interval uncertainty set, Ben-Tal & Nemirovski (1998, 1999) proposed using an ellipsoidal set which leads to solving the robust counterparts as a conic quadratic problem. The proposed uncertainty set (Ben-Tal & Nemirovski, 2000) involves a combined interval and ellipsoidal uncertainty set to ensure a less conservative robust model. Specifically, the uncertainty set can be described as follows:

\[\mathcal{E}_{int+ell}(1, 1.14)\]

(3)

(4)

(5)

(6)

Note that the parameter $\Omega$ is defined as $\Omega = (\| \bar{a}_j \| + 1)^{1/2}$ where $\Omega = (\| \bar{a}_j \|)^{1/2}$ is the highest protection (i.e., the highest ellipsoid containing the box, $\bar{a}_j / \| \bar{a}_j \| \leq y_j$) the decision maker can seek for the constraint. The probability of violation of this $\Omega$ constraint is bounded above by $e^{-\alpha^2\Omega}$ for any $\alpha^2$. As mentioned before, $\mathcal{E}_{int+ell}(1, \Omega)$ is defined over the $\ell_2$-norm. Therefore, the dual of the subproblem in model (2) involves quadratic functions $\left( \sum_{j=1}^m \bar{a}_j^2 x_j^2 \right)^{1/2}$ whose solution leads to second-order cone programming.⁴ The robust counterpart of the

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³ Soyster (1973) robust formulation is one of the first models to immunize column-wise uncertainty in LP where uncertain parameters are confined to a convex set. Though the resulting robust model is linear, however, by taking the worst-case value of each uncertain parameter in the set, the approach becomes too conservative, sometimes producing results worse than the nominal problem.

⁴ The main drawback of this formulation which makes it difficult to implement in practice is that it is computationally demanding since the robust counterpart is a nonlinear convex programming.
optimization problem over the ellipsoidal uncertainty set or its intersection with the interval can lead to computationally tractable solutions or NP-hard problem. For large scale problems where interior points techniques can be harnessed, for instance, the ellipsoidal uncertainty set leads to practically tractable conic quadratic programming solutions (Ben-Tal & Nemirovski, 1998; 1999).

2.3. Robust counterpart to combined interval and polyhedral uncertainty set

Bertsimas & Sim (2004) relied on the family of a polyhedral set to propose a new robust formulation. The authors formulated an uncertainty set known as the budget of uncertainty set or the Bertsimas & Sim (2004) uncertainty set which is the commonly used uncertainty sets in practice because of its advantage in preserving the linearity of the nominal problem. Ordinarily, the uncertainty set involves a combined interval and polyhedral set described as:

\[ U = \{ \alpha_{ij} \} \quad \text{where} \quad \alpha_{ij} = \{ a_{ij} \} \quad \text{with} \quad a_{ij} = \begin{cases} \alpha_{ij}^l & \text{if } i = 1, \ldots, n, j = 1, \ldots, m \\ \alpha_{ij}^u & \text{if } i = 1, \ldots, n, j = 1, \ldots, m \\ \alpha_{ij}^\alpha & \text{with } \alpha_{ij}^\alpha = \alpha_{ij}^l + (1 - \alpha)\alpha_{ij}^u & \text{with } \alpha \in [0, 1] \\ \end{cases} \]

For simplicity, we consider \( \Phi = 1 \) and denote \( \Phi_{ij} \) as the budget uncertainty set where \( \Gamma \) is the parameter which the decision-maker can trade-off robustness and performance. Fig. 1c illustrates the distribution of \( \Phi_{ij} \) in the combined uncertainty set \( \Phi_{ij} + \Phi_{ij} \). The parameter, \( \Gamma \) of the \( \Gamma \)-th constraint takes values in \([0, 1] \) and the robust solution is feasible if only less than \( \Gamma \) uncertain coefficients change. Besides, the uncertain parameters have maximum protection if at most \( \Gamma \) coefficient of the uncertain \( \Gamma \)-th constraints are allowed to deviate. Under this uncertainty set dynamics, the robust counterpart of the subproblem (2) to the uncertainty set \( \Phi_{ij} \) is the following:

\[
\text{max} \sum_{j=1}^{m} \alpha_{ij} x_j \\
\text{s.t.} \sum_{j=1}^{m} a_{ij} x_j + \beta(y, \Gamma) \leq b_i & \quad \forall i, j \in \mathcal{J} \\
-y_j \leq x_j \leq y_j & \quad \forall i, j \in \mathcal{J} \\
l_j \leq x_j \leq u_j & \quad \forall j \in \mathcal{J} \\
y_j \geq 0 & \quad \forall i, j \in \mathcal{J} 
\]

where \( \beta(y, \Gamma) = \max_{\{a_{ij} \in [a_{ij}^l, a_{ij}^u] \}} \left\{ \sum_{i=1}^{n} \alpha_{ij} y_i + (1 - \alpha)\alpha_{ij} y_i \right\} \) is the protection function of the \( \Gamma \)-th constraint and \( y = (y_1, \ldots, y_n) \in \mathbb{R}_{+}^n \). Moreover, since model (8) is nonlinear, by strong duality to the subproblem, Bertsimas & Sim (2004) showed that an equivalent robust linear optimization has the formulation,

\[
\text{max} \sum_{j=1}^{m} \alpha_{ij} x_j \\
\text{s.t.} \sum_{j=1}^{m} a_{ij} x_j + \beta(y, \Gamma) \leq b_i & \quad \forall i, j \in \mathcal{J} \\
l_j \leq x_j \leq u_j & \quad \forall j \in \mathcal{J} \\
y_j \geq 0 & \quad \forall i, j \in \mathcal{J} 
\]

The robust solution parameter \( \Gamma \) regulates the number of \( \alpha_{ij} \) that may deviate from its nominal and obstruct the objection function. The higher value of a chosen \( \Gamma \) indicates a higher protection for the constraint and vice versa. The probability for the violation of the \( \Gamma \)-th constraint is given by \( e^{-\Gamma^2/2} \).

3. Robust counterpart with nonnegative decision variables

In practice, a decision maker would prefer robust solutions for which the decision variable is positive. This condition is non-negotiable for many operations research problems such as robust efficiency scores via the data envelopment analysis, transportation problem and in some engineering and business applications. However, given that the unrestricted interval \( l_j \leq x_j \leq u_j \) in the GRC could assume negative bounds and subsequently negative value for \( x_j \). the goal of this section is to seek alternative reduced robust formulations that are restricted to the interval. We will compare robust optimization problems for the general LP model (1) with its reduced form when \( l_j = 0 \) and \( u_j = \infty \) for \( j = 1, \ldots, n \) (or equivalently when \( x_1, \ldots, x_n \geq 0 \)). We believe that taking the no-negativity constraints into consideration theoretically reduces the size of the corresponding robust counterpart optimization as well as practically decrease their required computational burden. To better understand the variations between the input sizes of the aforementioned robust counterparts, first, we look at the robust counterpart optimization confined to the positive half plane. As aforementioned, we shall call it the reduced robust counterpart (RRC). The RRC considers robust counterpart optimization for the nonnegative decision variable,
max $\sum_{j=1}^{m} c_j x_j$
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j + \max_{\eta_j \in \mathcal{E}} \left\{ \sum_{j \in \mathcal{E}} \tilde{a}_{ij} x_j \right\} \leq b_i, \quad i = 1, \ldots, m\]
\[x_j \geq 0, \quad j = 1, \ldots, n\]  \(\text{(10)}\)

Let $X^{\text{RC}}$ be the feasible region of model (10), i.e.,

\[X^{\text{RC}} = \left\{ x : a_1 x + \max_{\eta_j \in \mathcal{E}} \left\{ \sum_{j \in \mathcal{E}} \tilde{a}_{ij} x_j \right\} \leq b_i, \forall \ i, x \geq 0 \right\} \subseteq \mathbb{R}_+^n\]

Then $X^{\text{RC}} \subseteq X$. The following theorem argues the redundancy of some constraints in the GRC model in the positive half plane which of course, reduces the computational burden for the reduced robust counterpart.

**Theorem 1.** The tractable GRC constraint under nonnegativity condition, i.e.,

\[\text{max } \sum_{j=1}^{m} c_j x_j\]
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j + \sum_{j \in \mathcal{E}} \tilde{a}_{ij} x_j \leq b_i, \quad i = 1, \ldots, m\]
\[-y_j \leq x_j \leq y_j, \quad \forall \ i, \forall j \in \mathcal{E}\]
\[0 \leq x_j \leq u, \quad j = 1, \ldots, n\]
\[y_j \geq 0, \quad \forall \ i, \forall j \in \mathcal{E}\]  \(\text{(11)}\)

is equivalent to the following model:

\[\text{max } \sum_{j=1}^{m} c_j x_j\]
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j + \sum_{j \in \mathcal{E}} \tilde{a}_{ij} x_j \leq b_i, \quad i = 1, \ldots, m\]
\[0 \leq x_j \leq u, \quad j = 1, \ldots, n\]  \(\text{(12)}\)

**Proof.** Let $(x^*, y^*) \in \mathbb{R}^{n \times \mathcal{E} \cup \mathcal{I} \cup \{0\}}$ be an optimal solution of model (11). It is plain to verify that model (11) possesses alternative optimal solutions: consider the following set:

\[X' = \{(x^*, y^*) : c_j^* \leq y_j, \forall j \in \mathcal{E}\}\]

Clearly, $(x^*, y^*)$ is a feasible solution for model (11) and its objective function value is equal to the objective function value of the optimal solution $(x^*, y^*)$. As a result, $X'$ is the set of all alternative optimal solutions of model (11). We arrive at model (12) when we let $c_j^* = c_j^* \forall j \in \mathcal{E}$ (note that $c_j^* \leq y_j^*$) which completes the proof. \(\square\)

**Theorem 1** suggests fewer parameters in solving the robust counterpart with nonnegative decision variables. It is easy to see that, when the GRC involves positive decision variables, some variables and constraints become redundant which can be removed in order to reduce the extra computational effort without altering the optimal objective value.

To see the implied usage of this suggested idea, we provide propositions which suggest that the robust counterparts discussed in Section 2 can be reformulated with fewer decision variables and constraints. The proof to these propositions can be inferred directly from the **Theorem 1** and therefore omitted.

**Proposition 1.** With the box uncertainty set $\mathcal{U}$ defined in the presence of nonnegative decision variable $x$, model (4) can be equivalently expressed as:

\[\text{max } \sum_{j=1}^{m} c_j x_j\]
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j + \Phi \sum_{j \in \mathcal{E}} \tilde{a}_{ij} x_j \leq b_i, \quad i = 1, \ldots, m\]
\[x_j \geq 0, \quad j = 1, \ldots, n\]  \(\text{(13)}\)

The proof follows analogous reasoning from **Theorem 1** since both $\tilde{a}_{ij}$ and $\Phi$ are nonnegative parameters.

**Proposition 2.** Given that the interval and polyhedral uncertainty set are intercepted in the positive half plane, the RRC with nonnegative decision variable $x$ to model (9) can be written as:

\[\text{max } \sum_{j=1}^{m} c_j x_j\]
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j + p_i \sum_{j \in \mathcal{E}} q_{ij} \leq b_i, \quad i = 1, \ldots, m\]
\[p_i \geq 0, \quad \forall \ i, \forall j \in \mathcal{I}\]
\[q_{ij} \geq 0, \quad \forall \ i, \forall j \in \mathcal{E}\]
\[x_j \geq 0, \quad j = 1, \ldots, n\]  \(\text{(14)}\)

The proof follows an analogous reasoning from **Theorem 1**.

### 3.1. Complexity analysis

One of the main challenging questions in robust optimization relates to the structure and complexity of the robust counterpart towards different classes of uncertainty set, $\mathcal{U}$. For instance, for a given class of nominal problem and structured uncertainty set, what would be the complexity class of the corresponding robust problem? (Bertsimas et al., 2011). Computational complexity theories allow us to group optimization problems into different difficulty level based on the number of computational resources required to solve a problem. For a tractable robust counterpart, efficient running of an optimization algorithm depends on the model structure and the evaluation stages used in finding an optimal solution. Our concern here lies in the computational burden of the models and the iteration counts required in solving each of the robust models. Note that, the number of iterations or steps necessary to solve these problems depends on the input size of the problem and the algorithm used to solve the problem. Here we consider the simple method which is the most popular and effective for solving linear programming problems. More formally, for a given optimization problem in the standard form:

\[\text{max } \sum_{j=1}^{m} c_j x_j\]
\[\text{s.t. } \sum_{j=1}^{m} a_{ij} x_j = b_i, \quad i = 1, \ldots, m\]
\[x_j \geq 0, \quad j = 1, \ldots, n\]  \(\text{(15)}\)

reducing Bazaraa et al. (2010), the average complexity of the simplex algorithm requires roughly $n \log n$ to $3n$ iterations of order to solve a problem. In most applications, the sparsity of the matrix $A = [a_{ij}]$ may be exploited to obtain a more efficient solution by the algorithm. The analysis with the simplex algorithm excludes the robust counterpart to interval + ellipsoid uncertainty set since the problem requires a non-linear approach such as the interior point method. Let $k = \sum_{j=1}^{m} |I|$ represent the total number of uncertain data. The standard form of model

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**Table 1**

<table>
<thead>
<tr>
<th>Robust Counterparts (RC)</th>
<th>RC - Box uncertainty set</th>
<th>RC - Polyhedral + interval uncertainty set</th>
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</thead>
<tbody>
<tr>
<td>General RC</td>
<td>Additions ((m + 2(k + n))(k + n + 1))</td>
<td>Additions ((m + 3k + 2n)(m + 2k + n + 1))</td>
</tr>
<tr>
<td></td>
<td>Multiplications ((m + 2(k + n))(k + n) + (m + 3(k + n)) + 1)</td>
<td>Multiplications ((m + 3k + 2n)(m + 2k + n) + (2m + 5k + 3n + 1))</td>
</tr>
<tr>
<td>Reduced RC</td>
<td>Additions (m(n + 1))</td>
<td>Additions ((m + k)(m + k + n + 1))</td>
</tr>
<tr>
<td></td>
<td>Multiplications (mn + m + n + 1)</td>
<td>Multiplications (m(k + m + n) + (2m + 2k + n + 1))</td>
</tr>
</tbody>
</table>
however, involves \( m + 2(k + n) \) constraints and \((n + 2k) + (m + 2(k + n))\) nonnegative decision and slack variables. Table 1 summarizes the additions and multiplications involved in the iterations of the linear robust counterparts.

The number of operation required in each iteration for the GRC in models (4) and (9) and the RRC models (13) and (14) using the simplex algorithm can be seen in Table 1. Clearly, the standard form of the GRC model (4) involves \( m + 2(k + n) \) constraints and \((n + k) + m + 2(k + n)\) nonnegative decision and slack variables while, on the other hand, the associated standard form of the RRC model (13) requires \( m + n \) iterations. Again, we note that the GRC model (9) contains \( m + 3k + 2n \) constraints and \( m + n + 2k \) decision variables and \( m + 2n + 3k \) slack variables whereas the RRC (14) involves only \( m + k \) constraints and \( 2(m + k) + n \) variables in all their standard forms. In summary, Table 1 indicates a significant reduction in the number of operations particularly for large-scale problems involving only nonnegative decision variables using models (13) and (14). From Theorem 1, if \( n \) is significantly larger than \( m \), solving these robust counterparts would result in saving a considerable computer storage and time.

A further question however is, does the RRC models (13) and (14) generate equivalent optimal solution as its corresponding GRC models (4) and (9)? Indeed the robust counterpart to a nonnegative optimization problem and its reduced form yield an equivalent solution as demonstrated by the following simple example in Bazzara et al. (2010):

\[
\begin{align*}
\min z &= -2x_1 - 4x_3 - x_5 \\
\text{s.t.} & \\
2x_1 + x_2 + x_3 &\leq 10 \\
x_1 + x_2 - x_3 &\leq 4 \\
\quad x_1 &\leq 4 \\
\quad x_2 &\leq 6 \\
1 &\leq x_3
\end{align*}
\]

The nominal values are \((2, 1, 1)\) and \((1, 1, -1)\) which are the coefficients of the first and second constraint respectively. The model is a minimization problem and its optimal solution and objective function value is \((x^*_1, x^*_2, x^*_3) = (0.067, 6, 2.67)\) and \(z^* = -28\), respectively. Assume that the uncertain coefficients are 10% accurate approximations of the “true” vector of coefficients. Let \( J_1 = \{1, 3\} \) and \( J_2 = \{2\} \). The corresponding robust counterpart based on the GRC model (4) is obtained as follows:

\[
\begin{align*}
\min z &= -2x_1 - 4x_3 - x_5 \\
\text{s.t.} & \\
2x_1 + x_2 + x_5 + \Phi (0.2x_1 + 0.1y_1) &\leq 10 \\
x_1 + x_2 - x_3 + \Phi (0.4y_3) &\leq 4 \\
\quad x_1 &\leq x_1 &\leq 1 \\
\quad x_2 &\leq x_2 &\leq 1 \\
\quad x_3 &\leq x_3 &\leq 1 \\
0 &\leq x_4 &\leq 4 \\
0 &\leq x_5 &\leq 6 \\
1 &\leq x_6 &\leq 4 \\
1 &\leq x_7 &\geq 0 \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]

The above CCR model is a non-linear programming problem and the optimal solution is difficult to be obtained. Charnes et al. (1978) therefore linearized the model (19) to the following LP model:

\[
\begin{align*}
\tilde{\theta} &= \max \sum_{j=1}^{m} u_j y_j \\
\text{s.t.} & \\
\sum_{j=1}^{m} v_j x_{ij} &= 1 \\
\sum_{j=1}^{m} u_j y_j - \sum_{j=1}^{m} v_j x_{ij} &\leq 0 & j = 1, \ldots, n \\
v_j &\geq 0 & j = 1, \ldots, m \\
u_j &\geq 0 & r = 1, \ldots, s
\end{align*}
\]

4. Data envelopment analysis approach

This section employs the robust DEA as a demonstration of the proposed models in Section 3. First, we present a basic radial DEA model with an input orientation. Note that, a priori, the traditional DEA models use equality in the normalization constraint and can be formulated as inequality constraints (Toolo, 2014a; Gorissen et al., 2015). So, initially, we deal with the problem of equality constraint to nonnegative optimization problem and its reduced form yield an equivalent solution as demonstrated by the following simple example in Bazaraa et al. (2010):
θ = max \sum_{i=1}^{n} u_i x_{ij} + u_o
\text{s.t.}
\sum_{i=1}^{n} v_i x_{ij} = 1
\sum_{i=1}^{n} u_i x_{ij} - \sum_{i=1}^{n} v_i x_{ij} + u_o \leq 0 \quad j = 1, \ldots, r
v_i \geq 0 \quad i = 1, \ldots, m
u_i \geq 0 \quad r = 1, \ldots, s
u_o \text{ free in sign}

(21)

where the free variable \(u_o\) indicates the returns to scale (RTS) of the under-evaluation unit which could be constant, increasing or decreasing. The first constraint of both models are known as normalization constraint. As a matter of fact, the CCR model is a special case of the BCC model where the RTS is constant, i.e., \(u_o = 0\). The CCR and BCC models are solved \(n\) times, each instance producing an optimal value \(\theta^\ast\) to obtain the relative efficiency of all DMUs. These problems can be solved by available commercial software such as DEA solver, LINGO, Max DEA, GAMS\textsuperscript{8} etc.

**Definition 1.** DMU is BCC-efficient if \(\theta^\ast = 1\) and there exist at least one strictly positive optimal solution (i.e., \(\forall i, v^\ast_i > 0, \forall r, u^\ast_r > 0\)) otherwise it is BCC-inefficient.

### 4.2. Equality constraint in uncertain DEA

Let \(x_{ij}\) and \(y_j\) be the nominal values for the uncertain inputs and uncertain outputs data of each DMU and let \(\delta_x\) and \(\delta_y\) be the deviations from the nominal inputs and outputs, respectively. In robust optimization, most often than not, the true values revolve in an unequal and symmetric interval. Equality constraints containing uncertain parameters are therefore required to be in the inequality form since the equality constraints can restrict the feasibility region or sometimes lead to the infeasibility of the robust model (Ben-Tal et al., 2009; Gorissen et al., 2015). However, uncertainty analysis carried out in DEA include modeling uncertainty in the normalization constraint, which is equality constraints in model (21). In this case, though the DEA models are always feasible, they become infeasible for robust analysis. Unless the uncertain inputs and outputs are analyzed in the envelopment form of the CCR model (see Sadjadi et al., 2011) for example, these models become unsuitable for general robust efficiency measurement. Equality constraint containing uncertain parameters for general robust optimization problems have been analyzed in different ways in applications. See Gorissen et al. (2015) for a summary of alternative approaches. As mostly considered, in DEA, alternative formulation converting the equality to inequality constraint is proposed (Toloo, 2014a) for the CCR model. Here, suppose the normalization constraint is fixed at any other positive parameter, model (21) can be restated as:

\[
\max \sum_{i=1}^{n} u_i x_{ij} + u_o
\text{s.t.}
\sum_{i=1}^{n} v_i x_{ij} = t
\sum_{i=1}^{n} u_i x_{ij} - \sum_{i=1}^{n} v_i x_{ij} + u_o \leq 0 \quad j = 1, \ldots, r
v_i \geq 0 \quad i = 1, \ldots, m
u_i \geq 0 \quad r = 1, \ldots, s
u_o \text{ free in sign}
\]

where \(t\) is a positive parameter.

**Remark 1.** It can easily be verified that \((u^\ast, v^\ast, u^\ast_o)\) is an optimal solution for model (21) if and only if \((u^\ast, v^\ast, u^\ast_o)\) is an optimal solution for model (22).

**Theorem 2.** The following model is equivalent to the BCC model (21):

\[
\min \sum_{j=1}^{m} \lambda_j x_{ij}
\text{s.t.}
\sum_{j=1}^{m} \lambda_j x_{ij} \leq \delta x_{ij} \quad i = 1, \ldots, m
\sum_{j=1}^{m} \lambda_j y_j \geq y_j \quad r = 1, \ldots, s
\sum_{j=1}^{m} \lambda_j = 1
\lambda_j \geq 0 \quad j = 1, \ldots, r
\theta \text{ free is sign}
\]

and

\[
\min \theta
\text{s.t.}
\sum_{j=1}^{m} \lambda_j x_{ij} \leq \delta x_{ij} \quad i = 1, \ldots, m
\sum_{j=1}^{m} \lambda_j y_j \geq y_j \quad r = 1, \ldots, s
\sum_{j=1}^{m} \lambda_j = 1
\lambda_j \geq 0 \quad j = 1, \ldots, r
\theta \geq 0
\]

(24)

Let \(\theta^\ast\) be the optimal solution of the dual models. It is easy to see that \(\theta^\ast > 0\) in model (24) since \(\theta^\ast \leq 0\) would mean \(\lambda^\ast = 0\), in the first constraint and subsequently violate the convexity constraint \(\sum_{j=1}^{m} \lambda_j = 1\). \(\theta^\ast\) can be derived from the dual model.

\[
\theta^\ast = \frac{\sum_{j=1}^{m} \lambda^\ast_j x_{ij} - \sum_{j=1}^{m} \lambda^\ast_j y_j}{\delta x_{ij} - y_j} \quad i = 1, \ldots, m
\]

(25)

For more details about solving DEA models with GAMS we refer the reader to Toloo (2014a).
which is impossible. Therefore, models (24) and (25) are equivalent and their related primal models are also equivalent. □

**Definition 2.** Let the parameter \( t \) be fixed at 1, the appropriate DEA model for the robust optimization analysis is the following:

\[
\max z
\]

\[\text{s.t.}\]

\[
z \leq \sum_{i=1}^{m} u_i x_{iy} - u_0 \leq 0
\]

\[
\sum_{i=1}^{m} v_{ij} y_{ij} \leq 1
\]

\[
\sum_{i=1}^{m} u_i x_{ij} - \sum_{i=1}^{m} v_{ij} y_{ij} + u_0 \leq 0
\]

\( i = 1, \ldots, m \)

\( j = 1, \ldots, n \)

\( v_i \geq 0 \)

\( u_i \geq 0 \)

\( r = 1, \ldots, s \)

\( u_0 \) free in sign

(26)

**Definition 2** provides a way to analyze uncertainties in both input and output data. Usually, for the majority of the DEA literature, considering the difficulty of uncertainty in the normalization constraint, when uncertainty appears in the input data, an input-oriented model is adopted whereas the output-oriented model is adopted with uncertain output data (Wang & Wei, 2010). Practically, as the choice of DEA orientation is not the prerogative of the decision analyst but mostly by the organization’s choice of production process model (26) is very useful for measuring the robust efficiency with uncertain inputs and outputs data in either the input or output orientation model.

5. Robust data envelopment analysis

The robust optimization in DEA has been introduced by Sadjadi & Omrani (2008) based on the reviewed approaches in Section 2. Extension to other advanced DEA models and a varied application has since been made, particularly for energy efficiency measurement. We refer the reader to Sadjadi et al. (2011), Omrani (2013), Lu (2015), and Wu, Ding, Koubaa, Chaala and Luo (2017). The modelling of robust optimization in DEA follows three main approaches in literature: the robust approaches proposed by Mulvey, Vanderbei and Zenios (1995), Ben-Tal & Nemirovski (2000) and Bertsimas & Sim (2004).

The Bertsimas & Sim (2004) approach is employed in this study and as described in Section 2, the true values of the uncertain input and output data are expressed as \( x_i^* = x_i + \eta_i^1 x_{ij}^1 \) and \( y_j^* = y_j + \eta_j^2 y_{ij}^2 \) where the independent random variables \( \eta_i^1, \eta_i^2 \) and \( \eta_j^1, \eta_j^2 \) are respectively.

\( x_i^* \) and \( y_j^* \) are the budget of uncertain inputs and outputs data in their true values. For each input data \( x_{ij}, i \in I \) and output data \( y_j, r \in R \) the true values are modelled as variables \( x_i^* \) and \( y_j^* \).\( \eta_i^1 \) and \( \eta_j^2 \) taking values in the interval [-1, 1] and the maximum deviations are defined as \( \eta_i^1 = \epsilon_i x_i \) and \( \eta_j^2 = \epsilon_j y_j \). Note that \( \epsilon_i \) is the percentage of perturbation specifying the amount of deviation from the uncertain inputs and outputs data from their true values. For each input data \( x_{ij}, i \in I \) and output data \( y_j, r \in R \) the true values are modelled as variables \( \eta_i^1, \eta_j^2 \).

6. The new approach: reduced robust DEA (RRDEA)

Now, with a strong emphasis on the computational complexity pointed out in Section 3.1 and to formalize our argument in Theorem 1 with the DEA, a reduced robust DEA (RRDEA) model based on Proposition 2 is formulated for model (28). We suppose that all the uncertain inputs and outputs data are protected with allowable deviation up to \( \Gamma_i^1 \) and \( \Gamma_j^2 \) and that an uncertain input and output data of DMU changes from their nominal values by \( \Gamma_i^1 = [\gamma_i^1 - \Gamma_i^1] x_{ij} \) and \( \Gamma_j^2 = [\gamma_j^2 - \Gamma_j^2] y_{ij} \). Then from Section 2.3, the robust counterpart to the uncertain DEA (26) can be formulated as below:

\[
\max \sum_{i=1}^{m} u_i x_{iy} + u_0 + \beta(\omega, 0, \Gamma^1, \Gamma^2)
\]

\[\text{s.t.}\]

\[
\sum_{i=1}^{m} v_{ij} y_{ij} \leq 1
\]

\[
\sum_{i=1}^{m} u_i x_{ij} - \sum_{i=1}^{m} v_{ij} y_{ij} + u_0 \leq 0 \quad \forall j \in J \quad \forall r \in R
\]

\( v_i \geq 0 \)

\( u_i \geq 0 \)

\( r = 1, \ldots, s \)

\( u_0 \) free in sign

where \( \omega \) is the efficiency score of DMU; \( \Gamma_i^1 \) and \( \Gamma_j^2 \) are non-negative variables of the set of uncertain inputs and outputs respectively; \( \gamma_i^1 \) and \( \gamma_j^2 \) are auxiliary variables of the absolute input and output weights; and \( \Gamma_i^1 \) and \( \Gamma_j^2 \) are the respective robust parameters of the uncertain inputs and outputs of the DMU under evaluation.
\[ \beta(\mathbf{u}', \mathbf{0}_m, \Gamma^*) = \max_{(s_{ij}[0] \leq \mathbf{u}_i \leq s_{ij}[0], [s_{ij}[1] \geq \mathbf{u}_j \leq s_{ij}[1])] \left\{ \sum_{i \in [j]} u_i Y_{ij} + (\Gamma^j - [\Gamma^j]) u_i Y_{ij} \right\} \]  

(31)

An analogous definition can be made for \( \beta(0, \mathbf{v}', \Gamma^*) \). Note that \( \Gamma^j = [\mathbf{l}] \) and \( \Gamma^j = [\mathbf{l}] \) lead to the worst-case formulation meanwhile \( \Gamma^* \) is chosen as integer numbers, then we obtain

\[ \beta(\mathbf{u}', \mathbf{v}', \Gamma^j + \Gamma^*) = \max_{(s_{ij}[0] \leq \mathbf{u}_i \leq s_{ij}[0], [s_{ij}[1] \geq \mathbf{u}_j \leq s_{ij}[1])] \left\{ \sum_{i \in [j]} u_i Y_{ij} + \sum_{i \in [j]} v_i Y_{ij} \right\} \]  

(32)

Again, notice that the above model is nonlinear. We make use of Theorem 1 and follow Theorem 3 below.

**Theorem 3.** The nonlinear model (29) is equivalent to the following reduced robust linear model:

\[
\begin{align*}
\text{max} & \quad \mathbf{w} \\
\text{s.t.} & \quad \mathbf{w} - \sum_{i=1}^{m} u_i x_{i0} + \sum_{i \in [j]} q_{ij} + \mathbf{u}_0 \leq 0 \\
& \quad \sum_{i=1}^{m} v_i x_{i0} + \sum_{i \in [j]} w_{ij} \leq 1 \\
& \quad \sum_{i=1}^{m} u_i x_{ij} - \sum_{i=1}^{m} v_i x_{ij} + \mathbf{p}^j y_{ij}^* + \mathbf{p}^j y_{ij}^\prime \leq 0, \quad j = 1, \ldots, n \\
& \quad \sum_{i \in [j]} q_{ij} + \sum_{i \in [j]} w_{ij} \leq 0, \quad \forall j, \forall r \in R_j \\
& \quad p^j_r + q_{ij} \geq u_i Y_{ij}, \quad \forall j, \forall r \in R_j \\
& \quad p^j_r + w_{ij} \geq v_i Y_{ij}, \quad \forall j, \forall i \in [j], \forall r \in R_j \\
& \quad p^j_r, q_{ij}, w_{ij} \geq 0, \quad \forall j, \forall i \in [j], \forall r \in R_j \\
& \quad q_{ij}, w_{ij} \geq 0, \quad j = 1, \ldots, n \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m \\
& \quad u_i \geq 0, \quad r = 1, \ldots, x \\
& \quad u_0 \text{ free in sign} \\
\end{align*}
\]

(33)

**Proof.** The protection function used in model (29) provides a simple way to generate a corresponding optimization problem for the input and output parameters. Given the optimal solution vector \((\mathbf{u}', \mathbf{v}')\), the protection function \( \beta(\mathbf{u}', \mathbf{v}', \Gamma^* + \Gamma^*) \) can be formulated as the following linear optimization problem:

\[
\begin{align*}
\text{min} & \quad \sum_{i \in [j]} q_{ij} + \sum_{i \in [j]} w_{ij} + p^j y_{ij}^* + p^j y_{ij}^\prime \\
\text{s.t.} & \quad p^j_r + q_{ij} \geq z^j, \quad \forall r \in R_j \\
& \quad p^j_r + w_{ij} \geq z^j, \quad \forall r \in R_j \\
& \quad z^j \leq u_i Y_{ij}, \quad \forall r \in R_j \\
& \quad z^j \leq v_i Y_{ij}, \quad \forall i \in [j], \forall r \in R_j \\
& \quad q_{ij}, w_{ij} \geq 0, \quad \forall i \in [j], \forall r \in R_j, R_j \\
& \quad q_{ij}, w_{ij} = 0, \quad j = 1, \ldots, n \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m \\
& \quad u_i \geq 0, \quad r = 1, \ldots, x \\
& \quad u_0 \text{ free in sign} \\
\end{align*}
\]

(34)

Model (34) is feasible and bounded for all \( \Gamma^j \) taking values in \([0, |I|]\) and \( \Gamma^j \) taking values in \([0, |R|]\). Therefore, by the strong duality theory, the dual model (35) is also feasible and bounded and their objective function values are equal. From Theorem 1, since \( u^*_i \) and \( v^*_i \) are nonnegative, model (35) can be simplified as model (36).

\[
\begin{align*}
\text{min} & \quad \sum_{i \in [j]} q_{ij} + \sum_{i \in [j]} w_{ij} + p^j y_{ij}^* + p^j y_{ij}^\prime \\
\text{s.t.} & \quad p^j_r + q_{ij} \geq z^j, \quad \forall r \in R_j \\
& \quad p^j_r + w_{ij} \geq z^j, \quad \forall i \in [j], \forall r \in R_j \\
& \quad q_{ij}, w_{ij} \geq 0, \quad \forall i \in [j], \forall r \in R_j, R_j \\
& \quad q_{ij}, w_{ij} = 0, \quad j = 1, \ldots, n \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m \\
& \quad u_i \geq 0, \quad r = 1, \ldots, x \\
& \quad u_0 \text{ free in sign} \\
\end{align*}
\]

(35)

Model (30) is equivalent to model (35) and by extension to extension (36), substituting model (36) into model (29) yields the resulting linear programming problem.

**Remark 4.** Model (33) should be very relevant for large-scale problems since a number of variables and constraints can be reduced to aid efficient computation in less time.

7. Application to banking data

This section demonstrates the applicability of the RRDEA model using data from 250 banks in some 23 countries in the European Union. The section consists of two subsections. The performance of the RRDEA with respect to complexity is presented in Section 7.1 and compared with the RDEA model (28). Section 7.2 provides a robust efficiency ranking of the banks using the RRDEA model. The banks analyzed comprise a conglomerate of European banks headquartered in the European Union with subsidiaries operating across the globe. They include, in terms of assets some large banks such as BNP Paribas, Deutsche Bank AG, HSBC Bank plc, Barclays Bank Plc, Société Générale SA whose operation extends beyond domestic European market. The data also comprises some less market share banks operating in the domestic market or solely in their single country.

Two major approaches are used in determining input and output measures in banking studies: intermediary approach and production approach (Berger & Humphrey, 1997). The former perceive banks as intermediaries between investors and savers, mainly transmitting capital and labor (inputs) to loans and securities (outputs). The latter assumes banks as producers of services (number of transactions and documents processed) such as producing loans, taking deposit account services (outputs) by using physical inputs; labor, capital etc. (Fethi & Pasiouras, 2010). There is no general consensus on the best approach to use in literature. In order to select the most appropriate bank features, we follow Mostafa (2009) where 26 research papers done on the banking industry in different countries are surveyed. Fig. 2 summarizes the approach adopted in this study.

Reference is made to Toloo & Tichý (2015) on the percentage of frequent selection of these banking measures presented in Mostafa (2009). Generally, employees are considered as an input variable and reasonably as fixed input. However, deposit is treated differently in banking studies; whiles 15.38% of research papers considered deposit as an input usually under the production approach, 26.92% of the...
surveyed papers measure it as output with the intermediate approach.\textsuperscript{10} Table 3 shows the descriptive statistics for the input and output measures. All the inputs and outputs variables are measured in millions of Euros. Employees – measured as the number of banking professionals and the non-banking staff is given in actual figures. As a result, the raw data are scaled for uniformity and to reduce round-off errors in the DEA models from excessively large values (Thanassoulis, 2001).

In order to assess the performance and complexity of the RRDEA model compared to the RDEA model under the GRC, uncertainties compelling volatilities in banks specific variables were considered. Banking sector uncertainties may originate from forecast values of loans and deposit, missing values, and measurement errors, etc. A DMU \(j\) is classified as uncertain if any of its inputs or outputs data is uncertain. Now we could consider the robust approach of Bertsimas & Sim (2004) to select the appropriate robust parameter \(\gamma_j\). For each DMU with \(i \in I_j\) and \(r \in R_j\), the percentage of perturbation, \(\epsilon_j\) of the nominal data is set to 0.01 and 0.05. For the choice of appropriate robustness level, it suffices to select \(\gamma_j\) approximately at

\[
\gamma_j = 1 + F^{-1}(1-\epsilon_j)\sqrt{2}\]

and respectively so for \(\gamma_j^I\) and \(\gamma_j^O\) so that the probability of the \(j\)th constraint violation is less than 1%. Note that the parameter \(F\) denotes the cumulative distribution of the standard Gaussian variable and \(\gamma\) represents the sources of uncertainty of each \(j\)th constraint (Bertsimas & Sim, 2004). In this paper, since the variable employee is given as fixed, there are three sources of uncertainties arising from the inputs and five sources of uncertainties for the outputs measures. To ensure full protection, \(\gamma_j\) is set to 8 and \(\gamma_j^I = 3\) and \(\gamma_j^O = 5\) which implies that the uncertain parameters are protected 100\% taking their worst-case value in the uncertainty set.

7.1. Performance of the reduced robust DEA

As with Table 1, a computational comparison of the iterations counts of models under the GRC and RRC is conducted with the 250 DMUs. The goal of the comparison is to understand the numerical differences in the computational complexity of the RDEA and RRDEA as the size of the problem increases. To do this, we consider five independent groups including 50, 100, 150, 200, and 250 DMUs. The total number of iterations for each group is obtained by running the robust models (28) and (33) with CPLEX solver in GAMS. Table 4 shows the groups and average runtime result for each group. There are significant differences between the iteration used by the two models in solving the same problem.

For instance, suppose we run model (28) with 200 DMUs, the RDEA model requires 1103 iterations to generate the robust efficiency scores. On the other hand, the RRDEA model (33) requires only 614 iterations for the same number of DMUs, showing 44.3\% reduction in the iterations count. It can be observed from column 3 that the robust model of under the GRC increases rapidly in the number of iterations as the number of DMU increases. This follows the exponential increase in the number of nonnegative variables and constraints that unwantedly increases the complexity of the model (De Klerk, 2008). Generally, from Theorem 1 it is noticeable that, particularly for large data set, the reduced robust model saves CPU time which by extension concludes a save in the computational cost of the reduced model to operations research problems with nonnegative decision variables.

7.2. Ranking of the European banks

The proposed robust approach is applied to the ranking of banks in Europe to demonstrate the applicability of the RRDEA to efficiency measurement. The same observation used for the first group of DMUs is considered. The result from the classical DEA model and the robust DEA models are reported in Table 5 which shows the efficiency scores and the rank of the DMUs (in bracket). A bank is efficient and operates on the efficient frontier under the VRS technology if its efficiency score is one. These efficient banks are shown in the 2nd column of Table 5 by

---

\textsuperscript{10}These performance measures are known as flexible measures (see Toloo, 2012, 2014) or dual-role factors (see Toloo & Barat, 2015; Toloo, Keshavarz, & Hatami-Marbini, 2018)
notwithstanding the obtained result, the efficiency scores in both models decrease as the perturbation of the uncertain data increases (see Fig. 3). Subsequently, the number of efficient banks in column 2 reduces when we tradeoff optimality for performance. And so, only few banks are closed to efficient when perturbation of the uncertain data increases from 1% to 5%. The mean of the robust models at 1% and 5% perturbation is reported at 0.942, 0.861 respectively. For managers in the banking industry, this indicates a higher price to pay for robustness when uncertainty level increases (Bertsimas & Sim, 2004). Finally, per the output result not presented here, it is also observed that the execution time for solving each LP in the RDEA far exceeds the RRDEA.

8. Concluding remarks

Robust counterpart optimization providing a general solution for decision variables has been the traditional way to study problems in operations research involving data uncertainty. However, in most practical problems where decision variables are nonnegative, the existing robust models present ‘unwanted variables’ that consume computational space particularly for large data set. The goal pursued in this paper is to offer alternative robust counterparts with nonnegative decision variables. The paper proposes reduced robust counterpart that attempts to minimize problem complexity without altering the optimality of the original solution.

In the DEA, while the decision variables are nonnegative, we find that the initial authors who proposed the robust DEA (Sadeghi & Omarzadeh, 2008) and hence subsequent researchers consider the original formulation in robust optimization where the decision variables can be negative (free in sign). We have shown in this paper that, such formulation involves many redundant constraints and decision variables which significantly increases the complexity of the robust DEA models and, of course, the required space and time for running the models. Addressing first the issue of infeasibility of simultaneous uncertainties in the DEA normalization constraint, we adjust the equality constraint in the normalization of the multiplier DEA models to inequality in order to allow for feasible and simultaneous consideration of uncertainties in the inputs and outputs data. Our complexity analysis using data from 250 European banks indicate that the proposed reduced robust DEA model renders some variables and constraints redundant in the RDEA models and reduces significantly the complexity in solving the same problem. In addition, almost 50% of the average reduction in iterations is found to save computational space. The proposed model while saving computational cost with problems with nonnegative decision variables also preserve the optimality of the original solution. By extension, the reduced robust counterpart with nonnegative decision variables can be used in many operations research applications such as portfolio analysis, supply chain management, and banking industry. Similar applications using the reduced robust DEA for the output orientation model and other uncertainty sets can be considered for further research in the future.

9. Statement of contribution

This research studies the robust optimization problem with nonnegative decision variables. It proposes alternative robust counterparts with nonnegative decision variables – a reduced robust approach and subsequently a reduced robust DEA model which addresses some shortcoming in previous studies. Moreover, it significantly decreases

Under the robust technical efficiency assessment, considering the uncertainties in the data, we should know that the feasibility of the optimal solution as well as the execution time for the robust models, can be affected heavily by just a small perturbation of the data (See Ben-Tal & Nemirovski, 2000). The optimal solution decreases with each consideration of higher perturbation of the DEA input and output data. For this reason, the efficiency scores of the robust models are smaller than the efficiency of the classical DEA model (See Fig. 3). In this paper, the result of the robust efficiency is also reported for the full protection of uncertain inputs and outputs, i.e., \( \gamma_i = 8 \) and \( \gamma_i^* = 3 \) and \( \gamma_i^* = 5 \). The result compares the efficiencies of the RRDEA and RDEA models. The last two pair of columns which are labeled “Robust DEA” and “Reduced robust DEA” practically validate our approach and illustrate that the robust models yield the same efficiency result at 1% and 5% perturbations of the uncertain data. The goal of reducing variables and constraints is therefore achieved without altering the optimal value and the information contemplated in decision making.

Notwithstanding the obtained result, the efficiency scores in both models decrease as the perturbation of the uncertain data increases (see Fig. 3). Subsequently, the number of efficient banks in column 2 reduces when we tradeoff optimality for performance. And so, only few banks are closed to efficient when perturbation of the uncertain data increases from 1% to 5%. The mean of the robust models at 1% and 5% perturbation is reported at 0.942, 0.861 respectively. For managers in the banking industry, this indicates a higher price to pay for robustness when uncertainty level increases (Bertsimas & Sim, 2004). Finally, per the output result not presented here, it is also observed that the execution time for solving each LP in the RDEA far exceeds the RRDEA.

8. Concluding remarks

Robust counterpart optimization providing a general solution for decision variables has been the traditional way to study problems in operations research involving data uncertainty. However, in most practical problems where decision variables are nonnegative, the existing robust models present ‘unwanted variables’ that consume computational space particularly for large data set. The goal pursued in this paper is to offer alternative robust counterparts with nonnegative decision variables. The paper proposes reduced robust counterpart that attempts to minimize problem complexity without altering the optimality of the original solution.

In the DEA, while the decision variables are nonnegative, we find that the initial authors who proposed the robust DEA (Sadeghi & Omarzadeh, 2008) and hence subsequent researchers consider the original formulation in robust optimization where the decision variables can be negative (free in sign). We have shown in this paper that, such formulation involves many redundant constraints and decision variables which significantly increases the complexity of the robust DEA models and, of course, the required space and time for running the models. Addressing first the issue of infeasibility of simultaneous uncertainties in the DEA normalization constraint, we adjust the equality constraint in the normalization of the multiplier DEA models to inequality in order to allow for feasible and simultaneous consideration of uncertainties in the inputs and outputs data. Our complexity analysis using data from 250 European banks indicate that the proposed reduced robust DEA model renders some variables and constraints redundant in the RDEA models and reduces significantly the complexity in solving the same problem. In addition, almost 50% of the average reduction in iterations is found to save computational space. The proposed model while saving computational cost with problems with nonnegative decision variables also preserve the optimality of the original solution. By extension, the reduced robust counterpart with nonnegative decision variables can be used in many operations research applications such as portfolio analysis, supply chain management, and banking industry. Similar applications using the reduced robust DEA for the output orientation model and other uncertainty sets can be considered for further research in the future.

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the classical DEA model. There are several banks which are purely technical efficient as measured by the BCC model. Notice that the efficiency scores of the banks reduce in the case of the robust models.

Under the robust technical efficiency assessment, considering the uncertainties in the data, we should know that the feasibility of the optimal solution as well as the execution time for the robust models, can be affected heavily by just a small perturbation of the data (See Ben-Tal & Nemirovski, 2000). The optimal solution decreases with each consideration of higher perturbation of the DEA input and output data. For this reason, the efficiency scores of the robust models are smaller than the efficiency of the classical DEA model (See Fig. 3). In this paper, the result of the robust efficiency is also reported for the full protection

of uncertain inputs and outputs, i.e., \( \gamma_i = 8 \) and \( \gamma_i^* = 3 \) and \( \gamma_i^* = 5 \). The result compares the efficiencies of the RRDEA and RDEA models. The last two pair of columns which are labeled “Robust DEA” and “Reduced robust DEA” practically validate our approach and illustrate that the robust models yield the same efficiency result at 1% and 5% perturbations of the uncertain data. The goal of reducing variables and constraints is therefore achieved without altering the optimal value and the information contemplated in decision making.

Notwithstanding the obtained result, the efficiency scores in both models decrease as the perturbation of the uncertain data increases (see Fig. 3). Subsequently, the number of efficient banks in column 2 reduces when we tradeoff optimality for performance. And so, only few banks are closed to efficient when perturbation of the uncertain data increases from 1% to 5%. The mean of the robust models at 1% and 5% perturbation is reported at 0.942, 0.861 respectively. For managers in the banking industry, this indicates a higher price to pay for robustness when uncertainty level increases (Bertsimas & Sim, 2004). Finally, per the output result not presented here, it is also observed that the execution time for solving each LP in the RDEA far exceeds the RRDEA.

8. Concluding remarks

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<table>
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<tr>
<th>Bank</th>
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<th>Robust DEA</th>
<th>Reduced robust DEA</th>
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<td>0.979</td>
<td>0.979</td>
</tr>
<tr>
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<tr>
<td>6</td>
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<td>0.979</td>
<td>0.979</td>
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</tbody>
</table>

Table 5

Ranking of banks based on the robust models.
the computational burden and preserves the optimality of the original problem. A case study involving 250 European banks is taken to illustrate the potential application of the suggested robust approach.

Acknowledgments

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Appendix A. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.cie.2018.10.006.

References


