DOCTORAL THESIS

Credit Risk Management Based on Copula Functions

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STATEMENT

I hereby declare that I have developed the entire doctoral thesis including annexes myself. All sources or information have been indicated in the bibliography and were quoted appropriately throughout the doctoral thesis.

Ostrava, ........................

Signature  ........................
Abstract

Credit risk is the major risk faced by financial institutions, so the significance of the measuring and managing credit risk is obvious. With the rapid development of the financial globalization, credit risk become more diversified and complicated. The dependence structure of financial time series is usually nonlinear, asymmetric, and time-varying. Copula functions can well describe the dependence structure and tail dependence of financial time series, which makes them widely used in the multivariate distribution of portfolio credit risk.

The general objective of the doctoral thesis is to investigate the credit risk measurements combined with the copula functions. The first subgoal is to compare the original credit risk model and the credit risk model based on copulas; the second subgoal is to compare the ability of pricing of credit derivatives under copula framework; and the third subgoal is to extend the constant copulas to the dynamic copulas and verify the best copula for given portfolios.

There are mainly four chapters in the thesis. Chapter 2 is the basic description of the copula theory, including elliptical copulas, Archimedean copulas, calibration of the copula parameters, and dependence measures.

Chapter 3 presents the general classification of credit risk models, including the structural and reduced-form models, and the framework of the CreditMetrics™ model. The CreditMetrics™ model is applied to two different portfolios consisting of ten bonds traded on FSE from 9th October 2017 to 8th October 2018, one is a good-quality portfolio, and another is a credit-risky portfolio. We calibrate the parameters of copula functions to fit the data and obtain the correlation matrix based on copulas. The assumption of normality is also removed. The comparison of the VaR of two portfolios by the original CreditMetrics™ model and the CreditMetrics™ model based on copulas suggests that copula functions improve the accuracy of the CreditMetrics™ model for the credit-risky portfolio at any confidence levels.

Chapter 4 focuses on the CDO pricing in the semi-analytic approach under copula framework. We summarize the conditional probabilities of default and the approximating distributions of the portfolio loss associated with different factor copula models. Then we study how the tranche spreads of a CDO will change according to different correlations and recovery rates based on the multinomial Gaussian copula and find that the CDO spreads increase with an increase in correlation and with a decrease in recovery rate for the senior tranche, while the
converse is true for the equity tranche. As for the mezzanine tranche, it is similar to the equity tranche in the relationship between the spreads and correlations, while it is similar to the senior tranche in the relationship between the spreads and recovery rates. Besides, we compare the ability of selected factor copula models to fit the market quotes and correlation skew according to Dow Jones iTraxx Europe tranches with 5-year maturity from 5th of January 2015 to 10th of May 2016 and the NIG model with lowest absolute error sum is the best.

Chapter 5 introduces dynamic copula models, including multivariate copula-ARCH model, time-varying copula model, and MRS-copula-GARCH model. We compare the negative log-likelihood of static copulas, time-varying copulas, and MRS-copula-GARCH copulas considering two stocks issued by ExxonMobil and IBM in FSE from 11th of May 2009 to 15th of March 2019 and then find that MRS-copula-GARCH copulas usually perform best. Besides, we study the dependence structure of four constituents of the FSET 100 index from 6th of June 2013 to 13th of May 2019 by the time-varying copula-GARCH models and find that the time-varying SJC copula is the optimal copula model because it can provide lowest negative log-likelihood.

**Key words:** Credit risk, copulas, factor models, dependence, time series
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Chapter 1

Introduction

The rapid development of the financial globalization has made financial market around the world more liberate and open than ever and further on integrated into one as a whole, which deepens their interdependence. Capital flow becomes freer and quicker in the world, facilitating the scale and the efficiency of the financial market. However, the degree of opening and the operation mechanism of the financial market vary from different countries. The volatile of one financial market can easily and fast influence the whole financial market due to the financial contagion effect. Therefore, the financial risks faced by the financial institutions become more diversified and complicated.

Financial risks usually include credit risk, market risk, liquidity risk, and operational risk. Credit risk is the potential loss for a financial institution when the borrower fails to meet the obligations in accordance with agreed terms. It is the oldest and largest risk that most financial institutions are faced with. With the increasing attention to the significance of credit risk, there have been numerous studies in measuring and managing credit risk. Many researchers have been developed a series of effective quantitative models for credit risk. Nowadays, researchers are still focusing on the improvement of credit risk models in order to better quantify and then better manage credit risk.

1.1 Background and significance of the research

Credit risk is diversified and complicated, displaying nonlinear, asymmetric, time-varying and tail dependence modes. Therefore, the simple linear correlation based on the traditional assumption of the normal distribution is not sufficient and accurate enough to measure credit risk. Copula function is a useful tool to describe dependence structure and determine the higher order joint default probabilities. The
propose of the copula theory can be traced back to 1959 by Sklar to link the multivariate distribution and copula function according to the marginally uniform representation. One of the most influential papers on copulas in Finance is written by Embrechts, McNeil and Straumann (2002). Since then, there are an enormous number of papers studying the copula theory in Finance and Economics. Joe (1997) and Nelsen (2006) provide clear and detailed introductions to copulas and their mathematical and statistical elements, and Cherubini et al. (2004) applies copulas to mathematical finance and financial derivatives pricing.

Since that the fat tails and excess kurtosis in the distribution of a single random variable usually increases the probability of extreme events, the non-zero tail dependence of a portfolio also increases the probability of joint extreme events. Cherubini and Luciano (2001), Embrechts et al. (2003), and Embrechts and Høing (2006) study the Value-at-Risk (VaR) of portfolios based on copula functions. Moreover, Rosenberg and Schuermann (2006) uses the method of copulas to construct the joint risk distribution in integrated risk management for financial institutions by aggregating risk types, including market, credit, and operational risk. McNeil et al. (2015) provides a readable textbook of copulas and risk management.

There is a specialized financial market for the credit derivatives called the derivatives market. A typical credit derivative is the credit default swap (CDS) and a portfolio credit derivative is the collateralized debt obligation (CDO). The primary function of credit derivatives is the securitization of credit risk, which means transforming credit risk into securities bought or sold by investors. The derivatives market reached a peak in the period leading up to the 2007-2009 global credit crisis. Although financial activities on CDOs has slowed down after the crisis, CDOs and related financial products still remain significant roles in credit risk management. A recent textbook written by Cherubini et al. (2004) is the first book to address the mathematics of copula functions illustrated with financial applications, such as the derivative pricing and the credit risk analysis. CDO is typically priced under the framework of reduced form models associated with copula functions. The one factor Gaussian copula, proposed by Li (2000), has been proved a benchmark for CDO pricing in a semi-analytical approach. Other variations on factor copulas can be found in Andersen and Sidenius (2004), considering certain nonlinear factor structures, and in Rogge and Schönbucher (2003) and Laurent and Gregory (2005), presenting factor copulas for modeling times-to-default. Applications of copulas in other derivatives markets include Rosenberg (2003), Bennett and Kennedy (2004),
van den Goorbergh et al. (2005), Salmon and Schleicher (2006), Grégoire et al. (2008) and Taylor and Wang (2010).

Consider the fact that the conditional volatility of financial time series changes through time, it is necessary to develop the models that also allow the conditional copula to change through time, which is so-called the time-varying copula model. The autoregressive conditional heteroskedasticity (ARCH) model is proposed by Engle (1982) to describe the conditional second moment of the financial time series to reflect the time-varying and clustering fluctuation. Time-varying copula models are considered in Patton (2001, 2004 and 2006), Jondeau and Rockinger (2006), Ausin and Lopes (2010), Christoffersen et al. (2012) and Creal et al (2013). Manner and Segers (2011) and Hafner and Manner (2012) introduce the stochastic dynamic models, which are analogous to the stochastic volatility models in Shephard (2005). Manner and Reznikova (2012) brings together different specifications for copula models with time-varying dependence structure and compares the applicability of each particular model in different cases.

1.2 Objective and structure

The general objective of the doctoral thesis is to investigate the credit risk measurements combined with the copula functions. The first subgoal is to compare the original credit risk model and the credit risk model based on copulas; the second subgoal is to compare the ability of pricing of credit derivatives under copula framework; and the third subgoal is to extend the constant copulas to the dynamic copulas and verify the best copula for given portfolios.

The thesis consists of four chapters that concern topics on credit risk management based on copula functions. Chapter 2 basically describes the copula theory. Chapter 2.1 and 2.2 present two fundamental types copulas, namely elliptical copulas and Archimedean copulas, theoretically and graphically. Chapter 2.3 discusses how to fit a copula to data. There are three main methods to calibrate the copula parameters, including maximum likelihood estimator (MLE), Inference functions for margins (IFM), and canonical maximum likelihood (CML). Dependence concepts and measures are considered in Chapter 2.4. Linear correlation is the standard measure for describing the dependence between financial assets regardless of many limitations of normality. Two further classes of measure, namely rank correlations and tail dependence, are directly related to copulas. Rank correlations can be used to
Chapter 1. Introduction

calibrate copulas to data, while coefficients of tail dependence aim at measuring the dependence between the joint extreme values.

Chapter 3 is concerned with credit portfolios with a view of credit risk management issues and copula functions. The main theme of portfolio credit risk is modelling the dependence structure of the defaults in the portfolio. We start from the general classification of credit risk models in Chapter 3.1, namely the structural and reduced-form models. The structural models assume that defaults occur when the value of the firm falls below a certain default point and a certain recovery is paid, while the reduced-form models assume that defaults occur exogenously and are unpredictable, and a separately specified recovery is paid. Chapter 3.2 detailedly describes the CreditMetrics™ model proposed by J. P. Morgan in 1997, including risk management framework, credit quality correlation, and Monte Carlo simulation. In Chapter 3.3, the empirical study, we conduct two different portfolios of bonds traded on Frankfurt Stock Exchange (FSE), one is a high-quality portfolio with ten good-rating bonds, and another is a credit-risky portfolio with ten risky bonds. The VaR of two portfolios are calculated and compared both by the original CreditMetrics™ model and by the CreditMetrics™ model based on the copula functions at different confidence levels. We find the empirical evidence that copula functions improve the accuracy of quantifying the VaR and therefore measuring credit risk for the credit-risky portfolio especially when the confidence level is low.

Chapter 4 focuses on CDO pricing under copula framework. Copula models are specifically developed to model portfolio credit derivatives and the Gaussian copula is the most popular copula model and has become a market standard for pricing CDO tranches. In Chapter 4.1, we describe the structure and properties of a CDO. Chapter 4.2 and 4.3 discuss CDO pricing and model calibration in various factor copula models. The large homogeneous portfolio approximation (LHP) is introduced in Chapter 4.4. Chapter 4.5 is the empirical study. The first study is about how the tranche spreads of a CDO will change according to different correlations and recovery rates based on the multinomial Gaussian copula. The second study fits different copula models to price the CDO through the market quotes, including Gaussian copula, double t copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings (RFL), and Clayton copula. The abilities of different factor copula models to fit the market quotes and correlation skew are analyzed and compared in terms of the minimal absolute error sum. The three extensions of the Gaussian copula models outperform and the NIG model
can provide the lowest absolute error sum.

Chapter 5 considers dynamic copula models to describe the dependence structure of the nonlinear, asymmetric, and dynamic features of financial time series. Chapter 5.1 firstly introduces the multivariate copula-ARCH models and the extensions include copula-GARCH, copula-IGARCH, and copula-EGARCH models. In Chapter 5.2, we describe time-varying copula models to form a dynamic conditional joint distribution. The bivariate time-varying Gaussian copula model and the bivariate time-varying symmetrized Joe-Clayton (SJC) copula model are presented as two typical examples. Chapter 5.3 extends the topic of dynamic copulas to Markov regime switching (MRS)-copula-GARCH model to draw the probabilistic inference in the form of a nonlinear iterative filter. Chapter 5.3 provides three empirical studies of dynamic copula models. The first and second study is the comparison of constant copulas, time-varying copulas, and time-varying copulas with Markov regime switching by ranking their negative log-likelihood. We find that time-varying copulas with Markov regime switching can perform best in most cases, followed by time-varying copulas. The third study analyzes the dependence of constituents of the FTSE 100 Index by the time-varying copula-GARCH models. The performances of selected copula models are evaluated by the negative log-likelihood and a powerful test named the Hit test is applied. The results suggest that the time-varying SJC copula with the minimal negative log-likelihood is the best.
Chapter 2

Copulas and dependence

A copula uses a marginally uniform representation of a multivariate distribution function. The original definition can be found in Sklar (1959), while Nelsen (2006) provides an easier introduction and derives the copula from a $d$-dimensional function with certain properties. The fundamental definitions and notations related to copulas are referred to McNeil et al. (2015).

**Definition 2.1 (copula).** An $m$-dimensional copula is a distribution function on $[0, 1]^m$ with standard uniform marginal distributions.

Hence, a copula is a fixed distribution function $C : [0, 1]^m \rightarrow [0, 1]$. In other words, the copula is the joint cumulative function of a vector of random variables $(U_1, \cdots, U_m)$, namely $C(u_1, \cdots, u_m) = P(U_1 \leq u_1, \cdots, U_m \leq u_m)$, where $P(U_i \leq u_i) = u_i$ and $u_i \in [0, 1]$ for all $i$.

The central theorem is called Sklar’s Theorem, which provides a valuable result on the uniqueness of copulas for continuous random variables.

**Definition 2.2 (Sklar’s Theorem).** Let $F$ be a joint distribution function with margins $F_1, \cdots, F_m$. Then there exists a copula $C : [0, 1]^m \rightarrow [0, 1]$ such that, for all $x_1, \cdots, x_m$ in $\mathbb{R} = [-\infty, \infty]$, $F(x_1, \cdots, x_m) = C(F_1(x_1), \cdots, F_m(x_m)). \tag{2.1}$

If the margins are continuous, then $C$ is unique; otherwise $C$ is uniquely determined on $\text{Ran}F_1 \times \text{Ran}F_1 \times \cdots \times \text{Ran}F_m$, where $\text{Ran}F_i = F_i(\mathbb{R})$ denotes the range of $F_i$. Conversely, if $C$ is a copula and $F_1, \cdots, F_m$ are univariate distribution functions, then the function $F$ defined in (2.1) is a joint function with margins $F_1, \cdots, F_m$.

Sklar’s Theorem is first introduced in Sklar (1959). A brief proof of Sklar’s Theorem can be found in McNeil et al. (2015) and a full proof can be found in Schweizer and Sklar (1983) and Nelsen (2006).
2.1 Elliptical copula

There are two fundamental parametric families of copula functions, namely Gaussian copula and $t$ copula. These two copulas are also named elliptical copulas or implicit copulas and do not have simple closed forms.

**Definition 2.3 (Gaussian copula).** The Gaussian copula (or normal copula) $C^G_{\Sigma}$ with the correlation matrix $\Sigma$ is defined as

$$C^G_{\Sigma}(u_1, \cdots, u_m) := \varphi_m(\varphi^{-1}(u_1), \cdots, \varphi^{-1}(u_m); \Sigma),$$  \hspace{1cm} (2.2)

where $\varphi_m$ is the standard $m$-dimensional normal distribution function, $\varphi^{-1}$ is the inverse of the standard univariate normal distribution function. Moreover, for a bivariate Gaussian copula, namely two dimensions, we write

$$C^G_{\rho}(u_1, u_2) = \int_{-\infty}^{\varphi^{-1}(u_1)} \int_{-\infty}^{\varphi^{-1}(u_2)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp\left[-\frac{s^2 - 2\rho st + t^2}{2(1 - \rho^2)}\right] ds dt,$$  \hspace{1cm} (2.3)

where $\rho$ is linear correlation parameter and $|\rho| < 1$.

**Definition 2.4 ($t$ copula).** The $t$ copula $C^t_{\nu, \Sigma}$ with the correlation matrix $\Sigma$ and $\nu$ degrees of freedom is

$$C^t_{\nu, \Sigma}(u_1, \cdots, u_m) := t_m(t^{-1}_\nu(u_1), \cdots, t^{-1}_\nu(u_m); \nu, \Sigma),$$  \hspace{1cm} (2.4)

where $t_m$ is the $m$-dimensional t distribution function, $t^{-1}_\nu$ is the inverse of the univariate $t$ distribution with $\nu$ degrees of freedom. Similarly, for a bivariate $t$ copula, we write

$$C^t_{\nu, \rho}(u_1, u_2) = \int_{-\infty}^{t^{-1}_\nu(u_1)} \int_{-\infty}^{t^{-1}_\nu(u_2)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp\left[1 + \frac{s^2 - 2\rho st + t^2}{2(1 - \rho^2)}\right]^{-\nu+2} ds dt.$$  \hspace{1cm} (2.5)

We show 1000 simulated random variates from the bivariate Gaussian copula and the bivariate $t$ copula in figure 2.1 below. The first row presents Gaussian copula with the correlation coefficient equals to 0.7, 0.8, and 0.9, and the second row presents $t$ copula with the degree of freedom equals to 3 and the correlation coefficient equals to 0.7, 0.8, and 0.9. Figure 2.2 plots an example of the density contours of multivariate distributions defined with Gaussian copula and $t$ copula with both margins being standard normal.
FIGURE 2.1: Generation of 1000 random variates from Gaussian copula (first row) and $t$ copula (second row)

FIGURE 2.2: Contour plots of Gaussian copula (left) and $t$ copula (right)
2.2 Archimedean copula

Unlike elliptical copulas mentioned in the last subchapter, Archimedean copulas usually have simple closed forms and are important for modelling the portfolio credit risk. A full description can be found in Nelsen (2006) and McNeil et al. (2015). We will introduce three main Archimedean copulas, including Gumbel copula in Gumbel (1958), Clayton copula in Clayton (1978), and Frank copula in Frank (1979).

**Definition 2.5 (Archimedean copula).** Let the function $\phi : [0, 1] \rightarrow [0, \infty)$ be a continuous, convex, and strictly decreasing function, which is subject to $\phi(0) = \infty$ and $\phi(1) = 0$. The Archimedean copula based on the generator $\phi$ is defined as

$$C(u_1, \ldots, u_m) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_m)) = \phi^{-1}\left(\sum_{i=1}^{m} \phi(u_i)\right),$$  

(2.6)

where $\phi^{-1}$ is the pseudo-inverse function of $\phi$.

**Definition 2.6 (Gumbel copula).** Let $\phi(t) = (-\ln t)^{\theta}$, $\phi^{-1}(t) = \exp\left[(-t)^{\frac{1}{\theta}}\right]$, where $\theta \in [1, \infty)$. The Gumbel copula is defined as

$$C_{G}^{\theta}(u_1, \ldots, u_m) = \exp\left\{-\sum_{i=1}^{m} (-\ln u_i)^{\theta}\right\}. \quad (2.7)$$

**Definition 2.7 (Clayton copula).** Let $\phi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, $\phi^{-1}(t) = (1 + \theta t)^{-\frac{1}{\theta}}$, where $\theta \in (0, \infty)$. The Clayton copula is defined as

$$C_{\theta}^{Cl}(u_1, \ldots, u_m) = \left(\sum_{i=1}^{m} u_i^{-\theta} - n + 1\right)^{-\frac{1}{\theta}}. \quad (2.8)$$

**Definition 2.8 (Frank copula).** Let $\phi(t) = -\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$, $\phi^{-1}(t) = -\frac{1}{\theta} \ln \left[1 + e^t(e^{-\theta} - 1)\right]$, where $\theta \in \mathbb{R} \setminus \{0\}$. The Frank copula is defined as

$$C_{\theta}^{Fr}(u_1, \ldots, u_m) = -\frac{1}{\theta} \ln \left[1 + \prod_{i=1}^{m} \frac{(e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{m-1}}\right]. \quad (2.9)$$

The family of Frank copulas are only Archimedean copulas that satisfy the functional equation $C(u_1, u_2) = \hat{C}(u_1, u_2)$ for so-called radial symmetry. $\hat{C}$ is the survival copula for $C$. In the bivariate case, $\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$. The remarkable result is proved in Frank (1979) in details.
Figure 2.3: Generation of 1000 random variates from Gumbel copula (first row), Clayton copula (second row), and Frank copula (third row)

We show 1000 simulated random variates from Gumbel copula, Clayton copula, and Frank copula in figure 2.3. The first row presents Gumbel copula, the second row presents Clayton copula, and the third row presents the Frank copula with the parameter alpha equals to 2, 4, and 6 in three different columns respectively. Figure 2.4 plots an example of the density contours of multivariate distributions defined with Gumbel copula, Clayton copula, and Frank copula, all with margins being standard normal.

Figure 2.5 provides an illustration in the bivariate case for the Gumbel copula and exploits the fact that observations of $v \sim \hat{C}$ can be immediately obtained from observations of $u \sim C$ in arbitrary dimensions according to the relationship $v = 1 - u$. 
Chapter 2. Copulas and dependence

Figure 2.4: Contour plots of Gumbel (left), Clayton (center), and Frank copula (right)

(A) Scatter plot of Gumbel copula  
(B) Scatter plot of survival Gumbel copula

(C) Perspective plot of Gumbel copula  
(D) Perspective plot of survival Gumbel copula

Figure 2.5: Gumbel copula and survival Gumbel copula with $\theta = 2$
2.3  Fitting a copula to data

There are three main methods to calibrate the copula parameters, including maximum likelihood estimator (MLE), Inference functions for margins (IFM), and canonical maximum likelihood (CML). Both the MLE method and the IFM method assume that the margins are parametric, while the CML method is based on the nonparametric estimation. Then the copulas can be selected by Akaike and Bayesian information criteria (AIC and BIC).

2.3.1 Maximum likelihood estimator

For continuous random variables, the copula density is related to the density of the distribution $F$, denoted as $f$:

$$f(x_1, \cdots, x_d) = c_\theta(F_1(x_1), \cdots, F_d(x_d)) \cdot \prod_{j=1}^{d} f_j(x_j),$$  \hspace{1cm} (2.10)

where $c_\theta$ is the density of $C_\theta$ and $f_j$ is the density of $F_j$, $j \in \{1, \cdots, d\}$.

Based on Lehmann and Casella (1998), the parameter vector $\theta$ can be estimated by the maximum likelihood estimator (MLE) by maximizing the log-likelihood function expressed as

$$l(\theta) = \sum_{i=1}^{n} \ln c_\theta(F_1(X_{i1}), \cdots, F_d(X_{id})) + \sum_{j=1}^{d} \sum_{i=1}^{n} \ln f_j(X_{ij}).$$  \hspace{1cm} (2.11)

Hence, the maximum likelihood estimator is

$$\hat{\theta}_{MLE} = \max l(\theta).$$  \hspace{1cm} (2.12)

Assume a 2-dimensional Clayton copula with the parameter $\theta = 3$. The estimation of the Clayton copula can be fulfilled by the function fitMvdc in the R package copula from 1000 random independent observations from the defined distribution with the marginal parameters of $\mu = 0$, $\sigma = 1$, and $\lambda = 1$. The function mvdc is used to construct a multivariate distribution by copula and parametric margins.

```r
library(copula)
c <- claytonCopula(3)
mvdc <- mvdc(c, margins=c("norm","exp"), paramMargins=list(list(mean=0,sd=1),list(rate=1)))
set.seed(712)
n <- 1000
```
A more detailed output including standard errors for the marginal and copula parameter estimations can be obtained by using `summary` on the object returned by `fitMvdc`:

```
summary(mle)
```

There are mainly two drawbacks of the MLE method. On the one hand, the estimation of $\theta$ relies on the parametric assumptions made on $F_1, \cdots, F_d$, which means the estimation will be biased once the margins are partially misspecified. On the other hand, the computation of the maximization of the log-likelihood is quite challenging over a potentially high-dimensional parameter space.

### 2.3.2 Inference functions for margins

The expression of the log-likelihood function is composed by two positive terms: one involving the copula density and its parameters and the other involving the
margins and all parameters of the copula density. Therefore, Joe and Xu (1996) proposes a two-stage estimator known as the inference functions for margins (IFM) estimator to release the burden of complicated computation associated with the MLE method. Joe (1997) points out that the IFM method is highly more efficient compared with the MLE method. Joe (2005) provides the theoretical properties of the IFM method.

In the first step, the marginal parameter vector $\theta_1$ is estimated by performing the estimation of the univariate marginal distributions:

$$
\hat{\theta}_1 = \arg \max_{\theta_1} \sum_{j=1}^{d} \sum_{i=1}^{n} \ln f_j(X_{ij}; \theta_1).
$$

(2.13)

And then in the second step, the copula parameter vector $\theta_2$ is estimated by

$$
\hat{\theta}_2 = \arg \max_{\theta_2} \sum_{j=1}^{d} \ln c_{\theta}(F_1(X_{i1}), \ldots, F_d(X_{id}); \theta_2; \hat{\theta}_1).
$$

(2.14)

The IFM estimator is defined as the vector:

$$
\hat{\theta}_{IFM} = (\hat{\theta}_1, \hat{\theta}_2)'.
$$

(2.15)

Consider the same settings as in the example in the last subchapter, the estimation of copula parameters by the IFM method in R can be realized by the function `fitCopula` with argument `method="ml"`. Note that data need to be parametric pseudo-observations obtained from fitted parametric marginal distribution functions. Thus, the returned large-sample standard error will underestimate the true standard error because the unknown estimation errors for the margins are not taken into account.

```
U <- cbind(pnorm(X[,1],mean=mean(X[,1]),sd=sqrt((n-1)/n)*sd(X[,1])),pexp(X[,2],rate=1/mean(X[,2])))
ifm <- fitCopula(claytonCopula(),data=U,method="ml")
summary(ifm)
```

Call: `fitCopula(copula, data = data, method = "ml")`

Fit based on "maximum likelihood" and 1000 2-dimensional observations.

Clayton copula, dim. d = 2
Estimate Std. Error
alpha 3.062 0.117
The maximized loglikelihood is 637.7
Optimization converged
Number of loglikelihood evaluations: function gradient
5 5
2.3.3 Canonical maximum likelihood

An alternative way to avoid the misspecification of margins is the canonical maximum likelihood (CML) method, which is originally suggested in Oakes (1994), formalized in Genest et al. (1995) and subsequently expanded upon in Tsukahara (2005). The CML method is based on the empirical marginal transformation. The transformation aims to nonparametrically estimate the unknown univariate margins \( F_1, \ldots, F_d \) with the empirical distribution functions \( \hat{F}_1, \ldots, \hat{F}_d \) defined as

\[
\hat{F}_{n,j}(x) = \frac{1}{n+1} \sum_{i=1}^{n} 1\{X_{ij} \leq x\}, \quad x \in \mathbb{R},
\]  

(2.16)

where \( 1\{X_{ij} \leq x\} \) is the indicator function.

The estimated margins are typically used to form the sample

\[
\hat{u}_{i,n} = (\hat{F}_{n,1}(X_{i1}), \ldots, \hat{F}_{n,d}(X_{id})), \quad i \in \{1, \ldots, n\},
\]  

(2.17)

which is usually referred to as a sample of pseudo-observations from \( C \).

Denote \( R_{ij} \) as the rank of \( X_{ij} \) among \( X_{ij}, \ldots, X_{nj} \) in increasing order for any \( j \in \{1, \ldots, d\} \). Note that \( \hat{F}_{n,j} \) is different from the standard empirical cumulative distribution function in the scalar \( 1/(n + 1) \) to ensure that the transformed data lies in the interior of the unit interval \( (0,1)^d \), which means \( \hat{F}_{n,j} = R_{ij}/(n + 1) \). Therefore, the sample of multivariate scaled ranks:

\[
\hat{u}_{i,n} = \frac{1}{n+1}(R_{i1}, \ldots, R_{id}), \quad i \in \{1, \ldots, n\}.
\]  

(2.18)

The CML estimator is

\[
\hat{\theta}_{CML} = \arg \max \sum_{i=1}^{n} \ln c_{\theta}(\hat{u}_{i,1}, \ldots, \hat{u}_{i,d}; \theta).
\]  

(2.19)

Let’s continue assuming a 2-dimensional Clayton copula with the parameter \( \theta = 3 \) from 1 000 random independent observations. In this case, the method is \textit{mpl} to maximize pseudo-likelihood.
Then we select the copulas by computing Akaike and Bayesian information criteria (AIC and BIC, respectively) for all available copulas and the copula with the minimum value is chosen. For observations \( u_{i,j}, i \in \{1, \cdots, n\}, j \in \{1, 2\} \), AIC of a bivariate copula \( C \) with parameter(s) \( \theta \) is given by

\[
AIC := -2 \sum_{i=1}^{n} [c(u_{i,1}, u_{i,2} | \theta)] + 2k,
\]

where \( k = 1 \) for one-parameter copulas and \( k = 2 \) for two-parameter copulas. Similarly, BIC is given by

\[
BIC := -2 \sum_{i=1}^{n} [c(u_{i,1}, u_{i,2} | \theta)] + \ln(n)k.
\]

Note that if BIC is chosen, the penalty for two-parameter copulas is stronger. More details of AIC and BIC can be found in Akaike (1974) and Schwarz (1978). Burnham and Anderson (2002) provides the model comparison by using these two criteria.

### 2.4 Dependence measures

There are several important types of dependence measure: the linear correlation, the Kendall’s tau, the Spearman’s rho, and the coefficients of tail dependence. Except the linear correlation belongs to the ordinary correlation, others are all copula-based. The definition of tail dependence can be referred to Joe (1993 and 1997). There are numerous alternative definitions of measures of tail dependence, such as in Coles et al. (1999). Essential books that describe the dependence concepts and emphasize the relationship between the dependence and copulas include Joe (1997), Denuit et al. (2005) and Rüschendorf (2013).
2.4.1 Linear correlation

Linear correlation, also named Pearson’s correlation, is the most frequently used measure of dependence. Two variables are normally distributed, linearity, and homoscedasticity. The Pearson’s correlation coefficient $r$ for the vector of random variables $(X, Y)^T$ is defined as

$$ r = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_X\sigma_Y}, $$

(2.22)

where $\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X)\text{E}(Y)$ is the covariance of $(X, Y)^T$, $\text{Var}(X)$ and $\text{Var}(Y)$ are the variances of $X$ and $Y$, and $\sigma_X$ and $\sigma_Y$ are the standard deviation of $X$ and $Y$.

The coefficient $r$ takes values in the interval $[-1, 1]$. If two variables are independent, $r = 0$; if $|r| = 1$, two variables are perfectly linearly dependent. Note that the converse of the first half is false, namely the zero correlation of two variables fails to imply the independence of them. Although linear correlation is widely applied in practice, it is usually suitable for finite variables with elliptical distributions. When the variables are infinite or with fat-tailed distributions, linear correlation coefficient might be misleading and make no sense.

2.4.2 Kendall’s tau and Spearman’s rho

Unlike linear correlation, which depends on both the copula of a bivariate distribution and the marginal distributions, rank correlations are non-parametric scalar measures of dependence only depend on the copula of a bivariate distribution and look at the ranks of the variables. Besides, they are invariant under increasing transformations of two random variables.

**Definition 2.9 (Concordance).** Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two observations from the vector $(X, Y)^T$ of continuous random variables. If $(x - \tilde{x})(y - \tilde{y}) > 0$, $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ are concordant; if $(x - \tilde{x})(y - \tilde{y}) < 0$, they are discordant.

The two important symmetric dependence measures for concordance include Kendall’s tau and Spearman’s rho. They both take values from $[-1, 1]$ and give the value 0 for independence random variables. More details can be found in Kruskal (1958) and Joag-Dev (1984).

**Definition 2.10 (Kendall’s tau).** Kendall’s tau for the random vector $(X, Y)^T$ and an independent copy $(\tilde{X}, \tilde{Y})^T$, with the same distribution but independent of
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\((X, Y)^T\), is defined as

\[
\tau = P[(X - \tilde{X})(Y - \tilde{Y}) > 0] - P[(X - \tilde{X})(Y - \tilde{Y}) < 0].
\]  

(2.23)

If \((X, Y)^T\) is a vector of continuous random variables with copula \(C\), Kendall’s tau is given by

\[
\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1.
\]  

(2.24)

Definition 2.11 (Spearman’s rho). Spearman’s rho for the random vector \((X, Y)^T\) and independent copies \((\tilde{X}, \tilde{Y})^T\) and \((X', Y')^T\) is defined as

\[
\rho = 3 \{P[(X - \tilde{X})(Y - Y') > 0] - P[(X - \tilde{X})(Y - Y') < 0]\}.
\]  

(2.25)

If \((X, Y)^T\) is a vector of continuous random variables with copula \(C\), Spearman’s rho is given by

\[
\rho = 12 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 3.
\]  

(2.26)

Figure 2.6 presents Spearman’s rho and Kendall’s tau as functions of the correlation parameter \(\rho\) of a Gaussian copula \(C_{\rho}^n\). As shown on the left-hand side, \(\rho\) can be approximated quite well by Spearman’s rho in this case. The right-hand side shows the corresponding error including the absolute error bound of 0.0181 as given in McNeil et al. (2015).
2.4.3 Tail dependence

The tail dependence measures the dependence between the upper-right quadrant tails and lower-left quadrant tails of a bivariate distribution. It aims at measuring the extremal dependence, or in other words, the dependence between the extreme values. Similar to rank correlations, tail dependences only depend on the copula of two random variables with continuous marginal degrees of freedom.

**Definition 2.12 (Upper and lower tail dependence).** Let \((X, Y)^T\) be a vector of continuous random variables with marginal distribution functions \(F\) and \(G\) respectively. The coefficient of upper tail dependence of \((X, Y)^T\) is

\[
\lambda_U = \lim_{u \to 1^-} P[Y > G^{-1}(u) \mid X > F^{-1}(u)]
= \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \to 0^+} \frac{\hat{C}(u, u)}{u}.
\]  

(2.27)

Similarly, the coefficient of lower tail dependence of \((X, Y)^T\) is

\[
\lambda_L = \lim_{u \to 0^+} [Y \leq G^{-1}(u) \mid X \leq F^{-1}(u)] = \lim_{u \to 0^+} \frac{C(u, u)}{u}.
\]  

(2.28)

Note that \(\lambda_U \in [0, 1]\) and \(\lambda_L \in [0, 1]\). If \(\lambda_U \in (0, 1]\), there exists the upper tail dependence; if \(\lambda_U = 0\), there is no upper tail dependence.

Equations (2.27) and (2.28) imply that the coefficients of the upper and lower tail dependence for a radially symmetric copula are equal. We can take a \(t\) copula as an
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Figure 2.7: Coefficient of tail dependence of \( t \) copula as a function of the correlation parameter \( \rho \) (left) and the degree of freedom \( \nu \) (right)

For a bivariate \( t \) copula \( C_{\nu,\rho}^t \), the coefficient is

\[
\lambda = \lambda_U = \lambda_L = 2t_{\nu+1}(\sqrt{\frac{(\nu + 1)(1 - \rho)}{1 + \rho}}),
\]

where \( t_{\nu+1} \) denotes the distribution function of the \( t \) distribution with \( \nu + 1 \) degree of freedom. Figure 2.7 presents the coefficient of tail dependence \( \lambda \), by the function \( \lambda \) in R, as a function of \( \rho \) and \( \nu \). For the fixed and finite \( \nu \), the tail dependence increases as \( \rho \) increases; while for the fixed \( |\rho| < 1 \), the tail dependence decreases as \( \nu \) increases.

```r
# Coefficient of tail dependence as a function of rho
rho <- seq(-1,1,by=0.01)
nu <- c(2,4,7,Inf)
n.nu <- length(nu)
lam.rho <- sapply(nu,function(nu.) # (rho,nu) matrix
  sapply(rho,function(rho.) lambda(tCopula(rho.,df=nu.))[["lower"]]))
expr.rho <- as.expression(lapply(1:n.nu,function(j)
  bquote(nu==.(if(nu[j]==Inf) quote(infinity) else nu[j]))))
matplot(rho,lam.rho,type="l",lty=1,lwd=2,col=1:n.nu,xlab=quote(rho),ylab = quote(lambda))
legend("topleft",legend=expr.rho,bty="n",lwd=2,col=1:n.nu)

# Coefficient of tail dependence as a function of nu
nu. <- c(seq(2,12,by=0.2),Inf)
rho. <- c(-1,-0.5,0,0.5,1)
n.rho <- length(rho.)
lam.nu <- sapply(rho.,function(rh) # (nu,rho) matrix
  sapply(nu.,function(nu) lambda(tCopula(rh,df=nu))[["lower"]]))
expr <- as.expression(lapply(1:n.rho,function(j)
  bquote(rho==.(rho.[j]))))
matplot(nu.,lam.nu,type="l",lty=1,lwd=2,col=1:n.nu,xlab = quote(nu),ylab=quote(lambda))
legend("right",expr,bty="n",lwd=2,col=1:n.nu)
```
Chapter 3
Portfolio credit risk model and copulas

Credit risk is omnipresent in the financial institutions, especially in most banks and insurance companies. The importance of measurement, pricing, and management of credit risk is therefore self-evident. In order to precisely measure credit risk, many credit risk models come into being. Broadly speaking, credit risk models can be divided into the structural and reduced-form models. The structural model begins with a crucial model of Merton (1974), while the reduced-form model is proposed by Jarrow and Turnbull (1995).

In real financial markets, credit risk is usually associated with the portfolios consisting of credit-risky instruments, such as bonds, loans, and credit derivative products. The core issue lies in the analysis of the dependence structure of the default events in the credit portfolios and the estimation of the value-at-risk (VaR) at different confidence levels. There are four influential portfolio credit risk models for calculating the VaR, including J. P. Morgan’s CreditMetrics™ in Gupton et al. (1997), Credit Suisse Financial Product’s CreditRisk+ in CSFP (1997), McKinsey’s CreditPortfolioView in Wilson (1997a and 1997b), and KMV’s CreditPortfolioManager in McQuown (1997) and Crosbie (1999). More details and comparative analysis can be found in Gordy (2000), Crouhy et al. (2000), Resti and Sironi (2012), and McNeil et al. (2015).

Credit risk management is the first area of application of copulas in Finance, because there are a growing amount of empirical evidence that the dependence structure between the default events is non-normal. The evidence against the normality early starts from Mills (1927), followed by Erb et al. (1994), Longin and Solnik (2001), Ang and Chen (2002), Ang and Berkaert (2002), and Bae et al. (2003). Hull and White (1998) is almost the earliest paper on VaR for collections of non-normal variables. Frey and McNeil (2001) points out that the asset correlations are not enough to describe dependence between defaults and it is the copula of the latent variables
that determines the higher order joint default probabilities for obligors, and thus determines the extreme risk in the portfolio. Cherubini and Luciano (2001), Embrechts et al. (2003) and Embrechts and Höing (2006) study the calculation of VaR of portfolios by using the copula methods. McNeil et al. (2015) proves that the loss distributions implied by threshold models are very sensitive to the copula of the critical variables. Therefore, it is reasonable to calibrate the dependence models for the critical variables based on the concept of copulas.

In this chapter, we start from the description of credit risk models of a single obligor, namely the structural and reduced-form models. Then portfolio credit risk models come to the fore and the important issue of dependence between default events is concerned. The CreditMetrics™ model, an industry model based on credit ratings, is introduced in detail. In the empirical study, the CreditMetrics™ model is applied to two well diversified portfolios consisting of ten bonds traded on Frankfurt Stock Exchange (FSE) to estimate the VaR of the portfolio at different significance levels. One portfolio is of high quality with ten good-rating bonds, while another is of credit risky with ten risky bonds. After that, CreditMetrics™ is concerned with the copula functions to recalculate the correlation matrix and the real distribution of the portfolio value. The VaR of two different portfolios by both the original CreditMetrics™ model and the CreditMetrics™ model based on the copula functions are compared and analyzed.

### 3.1 Credit risk management

Credit risk models can typically be classified into the structural and reduced-form models. In general, these two models differ in the recovery assumption. The structural models, based on Black and Scholes (1973) and Merton (1974), assume that defaults occur when the value of the firm falls below a certain default point and a certain recovery is paid. While the reduced-form models, represented by Jarrow and Turnbull (1995) and Duffie and Singleton (1999), assume that defaults occur exogenously and are unpredictable, and a separately specified recovery is paid.

#### 3.1.1 Structural model

In the structural models, the default occurs whenever the asset value, usually represented by a stochastic variable or a stochastic process, falls below the threshold.
These models typically start from the Merton model.

The Merton model was originally proposed by Robert Merton in 1974. It assumes that the firm’s liabilities simply comprise the equity (i.e., an issuing stock) and the debt (i.e., a bank loan or zero-coupon bond) with maturity $T$. The value of the equity and the debt at time $t$ is $S_t$ and $B_t$. Moreover, the markets are frictionless, which means there are no taxes or transaction costs, the value of the firm’s assets is therefore the sum of the value of the equity and the debt, i.e., $V_t = S_t + B_t$. At the maturity, we summarize the value of the equity and the debt in terms of the payoff of a European call and put option respectively:

$$S_T = \max(V_T - B, 0) = (V_T - B)^+, \quad (3.1)$$

$$B_T = \min(V_T, B) = B - (B - V_T)^+. \quad (3.2)$$

In the Merton model, the asset value process $V_t$ is assumed to obey a standard geometric Brownian motion (GBM):

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \forall \mu \in \mathbb{R}, \sigma > 0, \quad (3.3)$$

where $\mu$ is the asset drift, $\sigma$ is the asset volatility, and $W_t$ is a standard Brownian motion. Based on Itô’s Lemma, (3.3) implies that

$$V_T = V_0 \exp\left((\mu - \frac{1}{2} \sigma^2)T + \sigma W_T\right), \quad (3.4)$$

where $W_T \sim N(0, T)$ and $V_T \sim N(\ln V_0 + (\mu - \frac{1}{2} \sigma^2)T, \sigma^2 T)$. The probability of default can be computed by

$$P(V_T \leq B) = P(\ln V_T \leq \ln B) = \Phi\left(\frac{\ln \left(\frac{B}{V_0}\right) - (\mu - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}}\right). \quad (3.5)$$

However, there are various strict assumptions in the Merton model. For example, the defaults only occur at the maturity, which is obviously too simplistic and contrary to reality. Therefore, a rich literature extends the original Merton model by assuming that the firms can default at any time before the maturity once the asset-value process crosses a default threshold for the first time. This class of models are
so-called the first-passage-time model. Formally, the default time $\tau$ is defined as

$$\tau = \inf\{t \geq 0 : V_t \leq B\}. \tag{3.6}$$


### 3.1.2 Reduced-form model

In the reduced-form models, the default mechanism is unspecified and the default time is modelled as a non-negative random variable. In the literature, the reduced-form models can be divided into two approaches, the intensity-based approach and the hazard process approach, depending on whether the information of the default free assets is introduced or not. If the information is not introduced, it is the intensity-based approach; otherwise, it is the hazard process approach.

Jarrow and Turnbull (1995) originally proposes the reduced-form model, assuming that the default time $\tau$ is the first jump of a Poisson process with the default intensity $\lambda$ when $\lambda$ is a constant. $\tau$ is exponentially distributed with the parameter $\lambda$. Then Jarrow, Lando, and Turnbull (1997) extends the original reduced-form model and estimates the relationship between the credit ratings and the default probabilities by a discrete state space Markov chain specified by a transition matrix. When the default time is an exponential random variable, the default process is a Cox process. Lando (1998) generalizes the model of Jarrow, Lando, and Turnbull (1997) to allow for stochastic transition intensities between credit ratings by constructing a Cox process. Duffie and Singleton (1999) parameterizes the losses at default in terms of the fractional reduction in market value that occurs at default with Markov diffusion or jump-diffusion state dynamics. Extensions can be found in Becherer and Schweizer (2005). Jeanblanc, Yor, and Chesney (2009) provides the overall mathematical background.
3.2 CreditMetrics

The CreditMetrics™ model, proposed by the U.S. bank J. P. Morgan in 1997 originally, is a tool for estimating the distribution of changes in the market value of a portfolio of credit exposures based on the data for migration rates, default rates, and spreads of borrowers with various given rating categories. Figure 3.1 presents a step-by-step introduction of the CreditMetrics™ model reprinted from Gupton et al. (1997).

We will describe CreditMetrics™ at length in subsequent pages. Roughly speaking, there are three main parts in the description of the CreditMetrics™ model, including risk management framework, credit quality correlation, and interpretation and application of results. Before we start, it is essential to know what is value-at-risk (VaR), which can be referred to McNeil et al. (2015).

Given some confidence level \( \alpha \in (0, 1) \), value-at-risk (VaR) of a portfolio with loss \( L \) at the confidence level \( \alpha \) is given by the smallest number \( l \) such that the probability that the loss \( L \) exceeds \( l \) is no larger than \( 1 - \alpha \). Formally,

\[
VaR_\alpha(L) = \inf\{P(L > l) \leq 1 - \alpha\}. \tag{3.7}
\]

VaR is therefore a quantile of the loss distribution. Typical values for \( \alpha \) are \( \alpha = 0.95 \) or \( \alpha = 0.99 \). Figure 3.2 illustrates the notion of VaR. The probability density function of a loss distribution is a vertical line at the value of the 95% VaR. The 95% VaR is around 2.2, which indicates there is a 5% chance of loss at this amount.
3.2.1 Risk management framework

In general, there are four steps to calculate the credit risk for a portfolio by using CreditMetrics™: credit rating migration, calculation of the future value of a bond, derivation of the yield curves, and credit risk estimation.

3.2.1.1 Credit rating migration

CreditMetrics™ assumes that each asset has been assigned a rating either produced by rating agencies or referred to an internal rating, which is derived from the historical data of the bank itself. Besides, those rating categories are indicative of the default and migration probabilities for the subsequent year. Then, risk comes. It is important to estimate both the likelihood of default and the chance of migrating to any credit quality state at the same risk horizon. It is necessary to specify both the default possibility and the possibilities that companies in one category migrate to other no matter how many rating categories or how these categories are constructed.

3.2.1.2 Calculation of the future value of a bond

In this step, future values are determined at the risk horizon and there are usually eight revaluations in simple one-bond case because the future value should be calculated separately for each migration state. Moreover, the eight valuations can be
divided into two cases – one is in the event of default, and other is in the event of upgrades or downgrades.

In the first case (in the event of a default), the recovery rate is estimated depend on the seniority classification of the asset. Table 3.1 summarizes the recovery rates in the state of default. For instance, if a BBB bond is senior unsecured, its mean value in default is supposed to be 51.13% of its face value, and its standard deviation of the recovery rate is 25.45%.

<table>
<thead>
<tr>
<th>Seniority class</th>
<th>Mean (%)</th>
<th>Standard deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior secured</td>
<td>53.80</td>
<td>26.86</td>
</tr>
<tr>
<td>Senior unsecured</td>
<td>51.13</td>
<td>25.45</td>
</tr>
<tr>
<td>Senior subordinated</td>
<td>38.52</td>
<td>23.81</td>
</tr>
<tr>
<td>Subordinated</td>
<td>32.74</td>
<td>20.18</td>
</tr>
<tr>
<td>Junior subordinated</td>
<td>17.09</td>
<td>10.90</td>
</tr>
</tbody>
</table>

**Table 3.1: Recovery rates by seniority class (% of face value)**

Source: Gupten et al. (1997, pp. 26)

In the second case (in the event of upgrades or downgrades), the change in credit spread is estimated based on a straightforward future value bond revaluation. The future value of a bond can be calculated as

\[
FV = C + \frac{C}{(1 + r)^2} + \cdots + \frac{C + M}{(1 + r)^n},
\]

(3.8)

where \(C\) is coupon payment, \(n\) is number of payments, \(r\) is forward yield, \(M\) is value at maturity or par value, and \(C + M\) is nominal value.

### 3.2.1.3 Derivation of the yield curves

Inspired by Resti (2000), the yield curves for discounting the future cash flows can be derived from the transition matrix. Let \(T_V\) be the transition matrix, excluding the default probabilities, and \(t_d\) be the vector of default probabilities. Further consider a vector of transition probabilities from default to other states, all probabilities in the vector should be zero since no defaulted borrower can upgrade in the following year. Besides, the default probability of remaining default for a defaulted borrower is 100%. The collection of all elements can be given by

\[
T = \begin{bmatrix} T_V & t_d \\ 0' & 1 \end{bmatrix}.
\]

(3.9)
Then it is possible to specify a two-year transition matrix, which is calculated as the product of $T$ and $T$

$$T^2 = T \cdot T = \begin{bmatrix} T_0^2 & (1 + T_0)T_d \\ 0' & 1 \end{bmatrix}. \tag{3.10}$$

Similarly, the $n$-year transition matrix can be formulated as

$$T^n = \begin{bmatrix} T_0^n & \sum_{i=0}^{n-1} T_i T_d \\ 0' & 1 \end{bmatrix}. \tag{3.11}$$

The last column of the $n$-year transition matrix presents the default probabilities for all rating categories. For any rating categories $i$, $p_i^n$ is the default probability within the following $n$ years and $(p_i^n - p_i^{n-1})$ is the default probability in the $n^{th}$ year.

Now the yield curves can be determined based on the default probabilities. The forward curve can be derived from the spot curve under the assumption of no-arbitrage:

$$r^F_n = \frac{(1 + r_t)^t}{(1 + r_{t-1})^{t-1}} - 1, \tag{3.12}$$

where $r_t$ is the risk-free interest rate, such as LIBOR or EURIBOR of an interest rate swap.

The relationship between the one-year interest rate $r_i^1$ for the $i^{th}$-rated borrower and the one-year risk-free rate $r_i^F$ can be expressed as

$$(1 + r_i^1) \cdot (1 - p_i^1) + p_i^1 \cdot RR = 1 + r_i^F, \tag{3.13}$$

where $RR$ is the expected recovery rate.

Then the relationship between the two-year interest rate $r_i^2$ and the two-year risk-free rate $r_i^F$ is

$$(1 + r_i^1)^2 \cdot (1 - p_i^2) + p_i^1 \cdot RR \cdot \frac{(1 + r_i^F)^2}{(1 + r_i^1)^2} + (p_i^2 - p_i^1) \cdot RR = (1 + r_i^F)^2, \tag{3.14}$$
where \((p^*_2 - p^*_1)\) is the default probability in the 2\textsuperscript{nd} year. Therefore, the two-year interest rate \(r^*_2\) can be given by

\[
\begin{align*}
    r^*_2 &= \sqrt{\frac{(1 + r^F_2)^2 - p^*_1 \cdot RR \cdot \frac{(1 + r^F_2)^2}{1 + r^F_2} - (p^*_2 - p^*_1) \cdot RR}{(1 - p^*_2)} - 1.}
\end{align*}
\] (3.15)

Similarly, the \(n\)-year interest rate \(r^*_n\) is

\[
\begin{align*}
    r^*_n = (1 + r^F_n) \cdot \left\{\frac{1 - RR \cdot \sum_{j=1}^{n} \frac{p^*_j - p^*_j-1}{(1 + r^F_j)^2}}{1 - p^*_n}\right\}^{\frac{1}{n}} - 1.
\end{align*}
\] (3.16)

### 3.2.1.4 Credit risk estimation

The last step is to estimate the volatility or standard deviation of value due to the rating migration. According to what we have already obtained from previous two steps, it is able to obtain the likelihoods of all possible rating migration, including upgrades, downgrades, and default, and the distribution of value with each rating migration.

### 3.2.2 Credit quality correlation

Generally speaking, it is too simplistic and unrealistic to assume a zero correlation because there are several common factors (i.e., the economic cycle, movements in the interest rates, and shifts in prices of the commodity) that will contribute rating changes and defaults of companies. CreditMetrics\textsuperscript{TM} requires the joint likelihood of credit movements among obligors, which means estimating the credit quality correlation parameters.

Asset value model is based on the proposal that a company’s asset value drives its credit rating changes and defaults. Because the value of a company’s asset determines the ability to meet its obligations, and if the value of the company’s asset is too much low to meet its obligations, the company will default.

Now suppose there is a series of levels for asset value at the end of the period and suppose a BB-rated company whose assets are worth 100 million euro. As presented in figure 3.3, the asset value of a company in a specific year determines the credit rating of this company. Moreover, the greater the asset value, the higher the credit rating.
Let us parameterize the asset value process to model the change in company’s asset value to evaluate its credit rating, namely the percent changes in asset value are normally distributed, the mean is denoted by $\mu$, and the standard deviation is denoted by $\sigma$. Besides, given the fact that the value of $\mu$ will not influence the final result of the exposition, we can assume $\mu = 0$ to make it easier. Use $Z_{\text{Def}}$, $Z_{\text{CCC}}$, $Z_{B}$, etc. to satisfy the situations in which if $R < Z_{\text{Def}}$, the company will default; if $Z_{\text{Def}} < R < Z_{\text{CCC}}$, the company will be re-rated to CCC; if $Z_{\text{CCC}} < R < Z_{B}$, the company will be re-rated to B; and so forth.

Table 3.2 presents the transition probabilities of a BB-rated company, and the probability of each rating can be computed by

$$
P\{\text{Default}\} = P\{R < Z_{\text{Def}}\} = \Phi(Z_{\text{Def}}/\sigma), \tag{3.17}$$

$$
P\{\text{CCC}\} = P\{Z_{\text{Def}} < R < Z_{\text{CCC}}\} = \Phi(Z_{\text{CCC}}/\sigma) - \Phi(Z_{\text{Def}}/\sigma), \tag{3.18}$$

and so on, where $\Phi$ denotes the cumulative distribution for the standard normal distribution.

Similarly, it is easy to obtain the transaction probabilities and asset value thresholds for a single-A rated company as presented in table 3.3.

The evolution of the two joint credit rating assumes that the two asset returns are correlated and normally distributed, and there is only the specific correlation $\rho$. 

![Figure 3.3: Credit rating migration driven by BB company asset value](image-url)
Chapter 3. Portfolio credit risk model and copulas

Table 3.2: Transition probabilities and thresholds for a BB company
Source: Resti and Sironi (2012, pp.415)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability from the transition matrix (%)</th>
<th>Cumulative probability (%)</th>
<th>Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default</td>
<td>$\Phi(Z_{Def} / \sigma) = 1.06$</td>
<td>1.06</td>
<td>$-2.30\sigma$</td>
</tr>
<tr>
<td>CCC</td>
<td>$\Phi(Z_{CCC} / \sigma) - \Phi(Z_{Def} / \sigma) = 1.00$</td>
<td>2.06</td>
<td>$-2.04\sigma$</td>
</tr>
<tr>
<td>B</td>
<td>$\Phi(Z_B / \sigma) - \Phi(Z_{CCC} / \sigma) = 8.84$</td>
<td>10.90</td>
<td>$-1.23\sigma$</td>
</tr>
<tr>
<td>BB</td>
<td>$\Phi(Z_{BB} / \sigma) - \Phi(Z_B / \sigma) = 80.63$</td>
<td>91.43</td>
<td>$1.37\sigma$</td>
</tr>
<tr>
<td>BBB</td>
<td>$\Phi(Z_{BBB} / \sigma) - \Phi(Z_{BB} / \sigma) = 7.73$</td>
<td>99.16</td>
<td>$2.39\sigma$</td>
</tr>
<tr>
<td>A</td>
<td>$\Phi(Z_A / \sigma) - \Phi(Z_{BBB} / \sigma) = 0.67$</td>
<td>99.83</td>
<td>$2.93\sigma$</td>
</tr>
<tr>
<td>AA</td>
<td>$\Phi(Z_{AA} / \sigma) - \Phi(Z_A / \sigma) = 0.14$</td>
<td>99.97</td>
<td>$3.43\sigma$</td>
</tr>
<tr>
<td>AAA</td>
<td>$1 - \Phi(Z_{AAA} / \sigma) = 0.03$</td>
<td>100.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3: Transition probabilities and thresholds for an A company
Source: Resti and Sironi (2012, pp.416)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability from the transition matrix (%)</th>
<th>Cumulative probability (%)</th>
<th>Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default</td>
<td>$\Phi(Z_{Def}' / \sigma') = 0.06$</td>
<td>0.06</td>
<td>$-3.24\sigma'$</td>
</tr>
<tr>
<td>CCC</td>
<td>$\Phi(Z_{CCC}' / \sigma') - \Phi(Z_{Def}' / \sigma') = 0.01$</td>
<td>0.07</td>
<td>$-3.19\sigma'$</td>
</tr>
<tr>
<td>B</td>
<td>$\Phi(Z_B' / \sigma') - \Phi(Z_{CCC}' / \sigma') = 0.26$</td>
<td>0.33</td>
<td>$-2.72\sigma'$</td>
</tr>
<tr>
<td>BB</td>
<td>$\Phi(Z_{BB}' / \sigma') - \Phi(Z_B' / \sigma') = 0.74$</td>
<td>1.07</td>
<td>$-2.30\sigma'$</td>
</tr>
<tr>
<td>BBB</td>
<td>$\Phi(Z_{BBB}' / \sigma') - \Phi(Z_{BB}' / \sigma') = 5.52$</td>
<td>6.59</td>
<td>$-1.51\sigma'$</td>
</tr>
<tr>
<td>A</td>
<td>$\Phi(Z_A' / \sigma') - \Phi(Z_{BBB}' / \sigma') = 91.05$</td>
<td>97.64</td>
<td>$1.98\sigma$</td>
</tr>
<tr>
<td>AA</td>
<td>$\Phi(Z_{AA}' / \sigma') - \Phi(Z_A' / \sigma') = 2.27$</td>
<td>99.91</td>
<td>$3.12\sigma'$</td>
</tr>
<tr>
<td>AAA</td>
<td>$1 - \Phi(Z_{AAA}' / \sigma') = 0.09$</td>
<td>100.00</td>
<td></td>
</tr>
</tbody>
</table>

between the two asset returns. Therefore, the covariance matrix for the bivariate normal distribution can be computed as

$$
\Sigma = \begin{bmatrix}
\sigma^2 & \rho\sigma\sigma' \\
\rho\sigma\sigma' & \sigma'^2 
\end{bmatrix}.
$$

(3.19)

Then under the assumption that the correlation $\rho$ between the two asset returns is not equal to zero, the probability that both companies remain in their current credit rating (the asset return for the BB-rated company falls between $Z_B$ and $Z_{BB}$, while the asset return for the A-rated company falls between $Z_{BB}B'$ and $Z_A'$) can be
computed as

\[ P\{Z_B < R < Z_B B', Z_B BB' < R < Z_{A}'\} = \int_{Z_B}^{Z_B B'} \int_{Z_B B' B}^{Z_{A}'} f(r, r'; \Sigma) dr' dr, \]  
(3.20)

where \( f(r, r'; \Sigma) \) denotes the density function for the bivariate normal distribution with covariance matrix \( \Sigma \), and \( r \) and \( r' \) denote the values that the two asset returns may take on in the specific intervals.

### 3.2.3 Monte Carlo simulation

Monte Carlo simulations, developed by Stanislaw Ulam and John Von Neumann, are designed to estimate the parameters of a particular probability distribution from the historical data and then the extraction of \( N \) simulated values for the risk factors.

The Cholesky decomposition, also named Cholesky factorization, is used in the Monte Carlo simulations. As described in Resti and Sironi (2012), in the case of two variables only, \( A \) and \( B \), the covariance matrix can be decomposed into two triangular matrices \( A \) and \( A^T \) as

\[
\Sigma = \begin{bmatrix}
\sigma_A^2 & \sigma_{AB}^2 \\
\sigma_{BA}^2 & \sigma_B^2
\end{bmatrix} = \begin{bmatrix}
\sigma_A & 0 \\
\sigma_{AB} \sqrt{\sigma_B^2 - (\sigma_{AB}^2/\sigma_A)^2} & \sigma_B - (\sigma_{AB}^2/\sigma_A)^2
\end{bmatrix} \cdot \begin{bmatrix}
\sigma_A & \sigma_{AB}^2/\sigma_A \\
0 & \sqrt{\sigma_B^2 - (\sigma_{AB}^2/\sigma_A)^2}
\end{bmatrix} = AA'.
\]  
(3.21)

Similarly, the correlation matrix can be decomposed as

\[
\Sigma = \begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\rho & (1 - \rho^2)^{1/2}
\end{bmatrix} \cdot \begin{bmatrix}
1 & \rho \\
0 & (1 - \rho^2)^{1/2}
\end{bmatrix}.
\]  
(3.22)

The following equations are used to calculate single element in the Cholesky decomposition matrix of a portfolio with \( n \) assets:

\[
p_{ii} = (\sigma_{ii} - \sum_{k=1}^{i-1} p_{ki}^2)^{1/2}, \forall i = 1, 2, \ldots, n, \]  
(3.23a)

\[
p_{ij} = (\sigma_{ii} - \sum_{k=1}^{i-1} p_{ik} \cdot p_{jk}) \cdot p_{ii}^{-1}, \forall i, j = 1, 2, \ldots, n, \]  
(3.23b)

\[
p_{ij} = 0, \forall i > j, \]  
(3.23c)

where \( p_{ii} \) and \( p_{ij} \) are single element of the Cholesky decomposition matrix.
Although the analytical method is accurate, it is only suitable for a small portfolio. Monte Carlo simulation is therefore needed. This method is based on a simulation of a large number of scenarios. The process of simulation for estimating the distribution of portfolio values consists of the following steps:

1. Generate $N$ uniformly distributed random variables $\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{iN}$ ranging from 0 to 1, convert them into normally distributed variables $\xi_{i1}, \xi_{i2}, \cdots, \xi_{iN}$ by $\xi_i = \varphi^{-1}(\varepsilon_i)$, and combine them into a vector $\xi$;

2. Compute the Cholesky decomposition matrix $A$ based on the correlation matrix according to (3.22) and (3.23);

3. Transform the vector $\xi$ into the vector $x = \xi \cdot A$ containing possible scenarios for the correlated asset value returns;

4. Find the thresholds corresponding to various rating categories based on the probability transition matrix by the inverse of the normal distribution function, similar to what we did in table 3.2 and table 3.3;

5. Compare each value of the vector $x$ with the thresholds to determine the simulated rating category and then simulated future value;

6. Compute the future value of the portfolio in each scenario by summing up simulated future values of $n$ assets and generate the distribution of future value of the portfolio.

### 3.3 Empirical study

Consider a diversified portfolio consisting of bonds, the VaR of the portfolio can be obtained by the CreditMetrics™ model described in the last subchapter. Note that the equity returns can be considered as a proxy for the asset returns and used to estimate the correlation matrix. The correlation matrix of the portfolio is calculated based on the linear correlation. The implicit assumption is that the distribution of the portfolio value is normally distributed, the Monte Carlo simulation, therefore, is conducted based on the normal distribution.

In order to improve the accuracy of the original CreditMetrics™ model, we can select the appropriate copula function by the fitting algorithm for each pair of bonds in the portfolio and then estimate the correlation coefficient based on the selected...
copula. What’s more, the method will be the Spearman’s rho instead of the simple linear correlation. By this way, there will be a new correlation matrix of the portfolio which is different from the correlation matrix obtained by the original CreditMetrics™ model. Besides, we will try to find the real distribution of the portfolio value instead of assuming the normal distribution directly for the Monte Carlo simulation using the same uniformly distributed random variables to avoid the errors caused by the generation of different random variables.

We will conduct two different portfolios, one is a high-quality portfolio with ten good-rating bonds, and another is a credit-risky portfolio with ten risky bonds. The comparison of the VaR will be made for these two portfolios respectively.

### 3.3.1 Analysis of a high-quality portfolio

There is a high-quality portfolio that consists of ten different bonds traded on Frankfurt Stock Exchange (FSE) with a total nominal value of 10 million euro and each bond is represented equally in a nominal value of 1 million euro to avoid bias caused by high nominal values. In order to diversify the portfolio, these bonds are from different industries, including financial services, chemicals, retail, automotive, food, textiles, metals, electronic, telecommunications, and oil & gas. Besides, ratings and maturities are chosen differently for the diversification as well. Table 3.4 summarizes the basic information of the good-quality portfolio, including rating, coupon, nominal value, maturity, market price, and number of each bond.

<table>
<thead>
<tr>
<th>Name</th>
<th>Rating</th>
<th>Coupon</th>
<th>Nominal value</th>
<th>Maturity</th>
<th>Price</th>
<th>pcs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commerzbank</td>
<td>BBB+</td>
<td>1.67%</td>
<td>100 000 €</td>
<td>Jul-27</td>
<td>99.00%</td>
<td>10</td>
</tr>
<tr>
<td>BASF</td>
<td>A</td>
<td>2.50%</td>
<td>100 000 €</td>
<td>Jan-24</td>
<td>109.85%</td>
<td>10</td>
</tr>
<tr>
<td>Tesco</td>
<td>BB+</td>
<td>5.00%</td>
<td>50 000 €</td>
<td>Mar-23</td>
<td>109.30%</td>
<td>20</td>
</tr>
<tr>
<td>Volkswagen</td>
<td>A+</td>
<td>1.38%</td>
<td>200 000 €</td>
<td>Jan-25</td>
<td>98.38%</td>
<td>5</td>
</tr>
<tr>
<td>Nestle</td>
<td>AA</td>
<td>2.38%</td>
<td>200 000 €</td>
<td>Jan-22</td>
<td>97.22%</td>
<td>5</td>
</tr>
<tr>
<td>Adidas</td>
<td>BBB</td>
<td>1.25%</td>
<td>10 000 €</td>
<td>Oct-21</td>
<td>102.61%</td>
<td>100</td>
</tr>
<tr>
<td>ArcelorMittal</td>
<td>BBB-</td>
<td>6.13%</td>
<td>20 000 €</td>
<td>Jun-25</td>
<td>108.46%</td>
<td>50</td>
</tr>
<tr>
<td>Philips</td>
<td>A-</td>
<td>0.75%</td>
<td>100 000 €</td>
<td>May-24</td>
<td>99.53%</td>
<td>10</td>
</tr>
<tr>
<td>Vodafone</td>
<td>BBB+</td>
<td>1.50%</td>
<td>100 000 €</td>
<td>Jul-27</td>
<td>98.01%</td>
<td>10</td>
</tr>
<tr>
<td>Equinor</td>
<td>AA</td>
<td>2.45%</td>
<td>100 000 €</td>
<td>Jan-23</td>
<td>95.85%</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 3.4: Basic information of the high-quality portfolio

As shown in table 3.4, all bonds are denominated in euros (€) and the nominal values range from 10 000 € to 200 000 €. The number of bonds in the portfolio
Chapter 3. Portfolio credit risk model and copulas

is calculated by 1 million euro divided by the nominal value of each bond. Their ratings are provided by the rating agency Stand & Poor’s (S & P). The highest rating of these companies is AA, while the lowest one is BB+. These bonds are senior unsecured because they are issued by renowned companies in Europe. According to table 3.1, the recovery rate for senior unsecured debt is 51.13% and the loss given default is 48.87%.

The risk-free interest rates from 2018 to 2027 are derived from the EURIBOR with a maturity of 1 year on first day of the year from 2010 to 2019 by the linear regression. Then the forward rates can be calculated according to (3.12). The EURIBOR rates and forward rates from 2018 to 2027 are summarized in table 3.5.

<table>
<thead>
<tr>
<th>Year</th>
<th>2018</th>
<th>2019</th>
<th>2020</th>
<th>2021</th>
<th>2022</th>
<th>2023</th>
<th>2024</th>
<th>2025</th>
<th>2026</th>
<th>2027</th>
</tr>
</thead>
<tbody>
<tr>
<td>r_t^f</td>
<td>-0.186</td>
<td>-0.121</td>
<td>-0.056</td>
<td>0.009</td>
<td>0.074</td>
<td>0.139</td>
<td>0.204</td>
<td>0.269</td>
<td>0.334</td>
<td>0.399</td>
</tr>
<tr>
<td>r_t</td>
<td>-0.186</td>
<td>-0.056</td>
<td>0.074</td>
<td>0.204</td>
<td>0.334</td>
<td>0.465</td>
<td>0.595</td>
<td>0.725</td>
<td>0.856</td>
<td>0.986</td>
</tr>
</tbody>
</table>

**Table 3.5: EURIBOR rates and forward rates from 2018 to 2027 (%)**

Source: Euribor-rates.eu

3.3.1.1 Original CreditMetrics model

The correlation matrix of the portfolio in table 3.6 is obtained from the historical market prices of stocks. The selected time horizon of each trading day is from October 9th, 2017 to October 8th, 2018. The stock prices can be found in Appendix A.1. The lowest correlation coefficient is 0.015 between Commerzbank and Nestle, while the highest correlation coefficient is 0.600 between BASF and Philips. Although they are from different industries, the high correlation between BASF and Philips can be explained by their highly cooperation since 2006 within the OLED 2015 initiative of Germany’s Federal Ministry of Education and Research (BMBF).

It is necessary to derive the yield curves before we calculate the future value of each bond in the portfolio. The yield curves are derived from a series of multiannual transition matrices according to (3.11). The original transition matrix from Standard & Poor’s and the 10th year transition matrix are presented in Appendix A.2. Table 3.7 summarizes default probabilities from 2018 to 2027.

Then we can derive the yield curves from 2018 to 2027 according to (3.16), which are summarized in table 3.8 and figure 3.4. We can find that almost all curves are upward sloping except four worse curves from B+ to CCC, which is consistent with
Chapter 3. Portfolio credit risk model and copulas

### Table 3.6: Correlation matrix of the high-quality portfolio

<table>
<thead>
<tr>
<th></th>
<th>CB</th>
<th>BASF</th>
<th>Tesco</th>
<th>VW</th>
<th>Nestle</th>
<th>Adidas</th>
<th>AM</th>
<th>Phil.</th>
<th>VF</th>
<th>Equi.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB</td>
<td>1.00</td>
<td>0.38</td>
<td>-0.02</td>
<td>0.39</td>
<td>0.01</td>
<td>0.08</td>
<td>0.39</td>
<td>0.27</td>
<td>0.30</td>
<td>0.12</td>
</tr>
<tr>
<td>BASF</td>
<td>0.38</td>
<td>1.00</td>
<td>0.07</td>
<td>0.59</td>
<td>0.17</td>
<td>0.39</td>
<td>0.51</td>
<td>0.60</td>
<td>0.31</td>
<td>0.12</td>
</tr>
<tr>
<td>Tesco</td>
<td>-0.02</td>
<td>0.07</td>
<td>1.00</td>
<td>0.09</td>
<td>0.08</td>
<td>0.07</td>
<td>0.06</td>
<td>0.08</td>
<td>0.08</td>
<td>0.09</td>
</tr>
<tr>
<td>VW</td>
<td>0.39</td>
<td>0.59</td>
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### Table 3.7: Default probabilities from 2018 to 2027 (%)

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<th>2020</th>
<th>2021</th>
<th>2022</th>
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<td>56.91</td>
<td>56.98</td>
</tr>
</tbody>
</table>

The example given in Gupton et al. (1997, pp. 27). The difference of default probabilities \( p_i - p_{i-1} \) decreases through years for B+ to CCC as shown in table 3.7, the interest rate \( r_i \) therefore increases. The up-sloped yield curves indicate that yields on longer-term bonds may continue to rise, while the down-sloped yield curves suggest that yields on longer-term bonds may continue to fall.

Once the yield curves are known, we can obtain the future value of each bond in the portfolio according to (3.8) as summarized in table 3.9. Generally, the future values decrease with lower rating categories. With their given rating categories by Standard & Poor’s, the highest future value in the portfolio is 220 284 € from Nestle, while the lowest one is 10 368 € from Adidas.
Chapter 3. Portfolio credit risk model and copulas

<table>
<thead>
<tr>
<th></th>
<th>2018</th>
<th>2019</th>
<th>2020</th>
<th>2021</th>
<th>2022</th>
<th>2023</th>
<th>2024</th>
<th>2025</th>
<th>2026</th>
<th>2027</th>
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<td>-0.055</td>
<td>0.075</td>
<td>0.205</td>
<td>0.336</td>
<td>0.466</td>
<td>0.597</td>
<td>0.728</td>
<td>0.859</td>
<td>0.989</td>
</tr>
<tr>
<td>AA+</td>
<td>-0.185</td>
<td>-0.054</td>
<td>0.077</td>
<td>0.208</td>
<td>0.338</td>
<td>0.469</td>
<td>0.600</td>
<td>0.731</td>
<td>0.863</td>
<td>0.994</td>
</tr>
<tr>
<td>AA</td>
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<td>-0.054</td>
<td>0.077</td>
<td>0.207</td>
<td>0.338</td>
<td>0.469</td>
<td>0.601</td>
<td>0.732</td>
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<td>0.994</td>
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</table>

**Table 3.8:** Yield curves from 2018 to 2027 (%)

Now we can conduct the Monte Carlo simulation according to the process mentioned in subchapter 3.2.3 and firstly generate 10,000 uniformly distributed random variables for each bond in the portfolio. The Cholesky decomposition matrix from the correlation matrix in table 3.6 according to (3.22) and (3.23) is presented in table 3.10. The thresholds based on the probability transition matrix can be found in Appendix A.3. Then we can obtain the simulated rating categories and determine the simulated future value for each bond and each scenario referred to table 3.9. The future value of the portfolio for each scenario is therefore the sum of simulated future values of 10 bonds.

The last step is to obtain the distribution of the future values of the portfolio $FV_j$ in the simulation and the mean value of the distribution $E(FV)$. Then we can rewrite the portfolio values as the differences from the mean value $\Delta V_j = FV_j - E(FV)$ and sort them from the smallest to the largest. The VaR, given by $-\Delta V_j$, can be located at different confidence levels. Besides, we also can compute the VaR by installing and librarying the package CreditMetrics in R and using the function `cm.CVaR (M, lgd, ead, N, n, r, rho, alpha, rating)` named Credit VaR, where $M$ is one-year empirical migration matrix, $lgd$ is loss given default, $ead$ is exposure at default, $N$ is number of companies, $n$ is number of simulated random variables, $r$ is risk-free interest rate, $rho$ is correlation matrix, $alpha$ is confidence level, and $rating$ is rating of companies.
Figure 3.4: Yield curves from 2018 to 2027 (%)

Figure 3.5 summarizes the VaR at different confidence levels. It is clear that the VaR decreases sharply when the confidence level rises, ranging from 47 273 € to 432 €. When the confidence level is 99.9%, the portfolio loss will not exceed 47 273 € within the maturity. Similarly, when the confidence level is 99.5%, 99%, and 95%, the portfolio loss will not exceed 18 152 €, 9 088 €, and 432 € respectively. Different values of confidence level can be determined according to different objectives.

3.3.1.2 CreditMetrics model based on copulas

We can use the VineCopula package in R to select an appropriate bivariate copula by maximum likelihood estimation according to (2.11) and (2.12) through the function BiCopSelect. Note that we transfer the selection algorithm into the pseudo-observations in the interval of [0, 1] using the function pobs. Let us take the calculation of the correlation coefficient between Commerzbank and Tesco as an example. We can narrow the range of the possible families to select from for the simplification, which means we select the appropriate copula only from the basic elliptical and Archimedean copulas we introduced in the previous subchapter, namely Gaussian, t, Gumbel, Clayton, and Frank copula. The criterion for bivariate copula selection is AIC and BIC according to (2.20) and (2.21).

```r
library("readxl")
Commerzbank <- read_excel("CreditRisk_Copula.xlsx")[,1]
Tesco <- read_excel("CreditRisk_Copula.xlsx")[,3]"
### Table 3.9: Future values of bonds in the high-quality portfolio according to ratings (€)

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<tr>
<th></th>
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<th>BASF</th>
<th>Tesco</th>
<th>VW</th>
<th>Nestle</th>
<th>Adidas</th>
<th>AM</th>
<th>Phil.</th>
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<th>Equi.</th>
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</table>

### Table 3.10: Cholesky matrix of the high-quality portfolio

```r
library(VineCopula)
ui <- pobs(as.matrix(cbind(Commerzbank,Tesco))[,1])
ui2 <- pobs(as.matrix(cbind(Commerzbank,Tesco))[,2])
selectedCopula <- BiCopSelect(ui,ui2,familyset=0:5)

The fitting algorithm select a t copula for the input data. It estimates the rho is 0.04, degree of freedom is 5.29, and Kendall’s tau is 0.02. Then we can use the copula package to double check the parameters by the suggested copula.

```
It is good to see that the parameters of the fitted copula are nearly the same as those estimated by the function `BiCopSelect`. We then can have a look at the perspective plot and contour lines of the $t$ copula based on the input data using the functions `persp` and `contour` as shown in figure 3.6. The functions `dCopula` and `pCopula` represent the density and the distribution function respectively.

The last step is to build the copula obtain the correlation coefficient. The function `rCopula` generates the random variables. Therefore, the correlation coefficient between Commerzbank and Tesco is around 0.034 based on $t$ copula.
We can obtain all correlation coefficients in the similar way and table 3.11 is the correlation matrix of the portfolio based on copulas. Most of correlation coefficients are slightly higher than before. The highest correlation coefficient is 0.662 still between BASF and Philips. Besides, the lowest correlation coefficient is 0.034 between Commerzbank and Tesco, which is a negative value of -0.028 previously.

<table>
<thead>
<tr>
<th></th>
<th>CB</th>
<th>BASF</th>
<th>Tesco</th>
<th>VW</th>
<th>Nestle</th>
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<th>Phil.</th>
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<td>CB</td>
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<td>0.378</td>
<td>0.034</td>
<td>0.391</td>
<td>0.035</td>
<td>0.092</td>
<td>0.369</td>
<td>0.277</td>
<td>0.327</td>
<td>0.125</td>
</tr>
<tr>
<td>BASF</td>
<td>0.378</td>
<td>1.000</td>
<td>0.125</td>
<td>0.577</td>
<td>0.179</td>
<td>0.512</td>
<td>0.489</td>
<td>0.662</td>
<td>0.337</td>
<td>0.149</td>
</tr>
<tr>
<td>Tesco</td>
<td>0.034</td>
<td>0.125</td>
<td>1.000</td>
<td>0.122</td>
<td>0.085</td>
<td>0.576</td>
<td>0.111</td>
<td>0.158</td>
<td>0.124</td>
<td>0.151</td>
</tr>
<tr>
<td>VW</td>
<td>0.391</td>
<td>0.577</td>
<td>0.122</td>
<td>1.000</td>
<td>0.177</td>
<td>0.354</td>
<td>0.451</td>
<td>0.407</td>
<td>0.366</td>
<td>0.262</td>
</tr>
<tr>
<td>Nestle</td>
<td>0.035</td>
<td>0.179</td>
<td>0.085</td>
<td>0.177</td>
<td>1.000</td>
<td>0.168</td>
<td>0.084</td>
<td>0.267</td>
<td>0.122</td>
<td>0.061</td>
</tr>
<tr>
<td>Adidas</td>
<td>0.092</td>
<td>0.512</td>
<td>0.376</td>
<td>0.354</td>
<td>0.168</td>
<td>1.000</td>
<td>0.311</td>
<td>0.418</td>
<td>0.330</td>
<td>0.086</td>
</tr>
<tr>
<td>AM</td>
<td>0.369</td>
<td>0.489</td>
<td>0.111</td>
<td>0.451</td>
<td>0.084</td>
<td>0.311</td>
<td>1.000</td>
<td>0.378</td>
<td>0.275</td>
<td>0.174</td>
</tr>
<tr>
<td>Phil.</td>
<td>0.277</td>
<td>0.662</td>
<td>0.158</td>
<td>0.407</td>
<td>0.267</td>
<td>0.418</td>
<td>0.378</td>
<td>1.000</td>
<td>0.334</td>
<td>0.124</td>
</tr>
<tr>
<td>VF</td>
<td>0.327</td>
<td>0.337</td>
<td>0.124</td>
<td>0.366</td>
<td>0.122</td>
<td>0.330</td>
<td>0.275</td>
<td>0.334</td>
<td>1.000</td>
<td>0.165</td>
</tr>
<tr>
<td>Equi.</td>
<td>0.125</td>
<td>0.149</td>
<td>0.151</td>
<td>0.262</td>
<td>0.061</td>
<td>0.086</td>
<td>0.174</td>
<td>0.124</td>
<td>0.165</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 3.11: Correlation matrix of the high-quality portfolio based on copulas

Figure 3.7 is a pie chart summarizes the numbers of different copulas in the portfolio. There are 10 elliptical copulas and 35 Archimedean copulas. Most of them are the survival Gumbel copula and only 2 of them are the Gumbel copula.

Before the Monte Carlo simulation, it is necessary to find out the real distribution of the portfolio value, presented in Appendix A.4, by using the fitdistplus package.
Figure 3.7: Numbers of copulas in the high-quality portfolio

and the \textit{logspline} package. The function \texttt{descdist} can provide the descriptive parameters of an empirical distribution for non-censored data. The result illustrates that the distribution is positive skewed and platykurtic. Figure 3.8 is a skewness-kurtosis plot proposed by Cullen and Frey (1999). The blue point in the figure named \textit{Observation} is plotted close to the point of the normal distribution.

\begin{verbatim}
library(fitdistrplus)
library(logspline)
descdist(portfolio_value,discrete = FALSE)

summary statistics
------
min: 573.096  max: 644.344
median: 605.83
mean: 607.1859
estimated sd: 15.56576
estimated skewness: 0.03960546
estimated kurtosis: 2.573875
\end{verbatim}

Let’s fit a normal distribution and a Weibull distribution respectively by the function \texttt{fitdist} and then plot the fit for both of them as shown in figure 3.9 and 3.10. Although both distributions can fit the data well, the normal distribution is slightly better when we have a look at the density Q-Q plot because the normal distribution looks better at tails. Therefore, the best distribution to fit the portfolio value is the normal distribution.

\begin{verbatim}
fit.norm <- fitdist(portfolio_value,"norm")
plot(fit.norm)
fit.weibull <- fitdist(portfolio_value,"weibull")
plot(fit.weibull)
\end{verbatim}
Chapter 3. Portfolio credit risk model and copulas

Figure 3.8: Skewness-kurtosis plot

Figure 3.9: Plot of the normal distribution
Table 3.12 is the Cholesky decomposition matrix based on the correlation matrix of the portfolio based on copulas. Note that the Monte Carlo simulation is still conducted by the normally distributed random variables, because the real distribution of the portfolio value follows a normal distribution. What’s more, in order to avoid the errors caused by the generation of different random variables, we will continue use the same normally distributed random values as what we used in subchapter 3.3.1.1.

Then we can compute the VaR based on copulas. The values of confidence level include 99.9%, 99.5%, 99%, and 95%. Figure 3.11 compares the original VaR and the VaR based on copulas. The VaR is positively related with the correlation. As shown in figure 3.11, the VaR based on the copula functions are slightly higher than the original VaR at different confidence levels because most of the correlation coefficients in the correlation matrix based on the copula functions are higher than those in the original correlation matrix. The credit risk is underestimated in the original CreditMetrics™ model no matter at which confidence level for the given portfolio, which means the extremely values cannot be reflected precisely.
Consider another credit-risky portfolio with ten bonds traded on FSE as well. The total nominal value is 10 million euro and each bond is represented in a nominal value of 1 million euro. The industries of these ten bonds belong to are the same as what we set in the previous high-quality portfolio.

Table 3.13 summarizes the basic information of the credit-risky portfolio. It is obvious that the rating category ranges from BBB+ to B+, and there is no A-rated bonds in the portfolio. According to table 3.1, the recovery rate for senior subordinated debt is 38.52% and the loss given default is 61.48%. Besides, the spot rates and forward rates from 2018 to 2027 are presented in table 3.5.

### Table 3.12: Cholesky matrix of the high-quality portfolio based on copulas

<table>
<thead>
<tr>
<th></th>
<th>CB</th>
<th>BASF</th>
<th>Tesco</th>
<th>VW</th>
<th>Nestle</th>
<th>Adidas</th>
<th>AM</th>
<th>Phil.</th>
<th>VF</th>
<th>Equi.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>BASF</td>
<td>0.378</td>
<td>0.926</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Tesco</td>
<td>0.034</td>
<td>0.121</td>
<td>0.992</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>VW</td>
<td>0.391</td>
<td>0.464</td>
<td>0.053</td>
<td>0.793</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Nestle</td>
<td>0.035</td>
<td>0.179</td>
<td>0.063</td>
<td>0.976</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Adidas</td>
<td>0.092</td>
<td>0.516</td>
<td>0.065</td>
<td>0.035</td>
<td>0.675</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>AM</td>
<td>0.369</td>
<td>0.378</td>
<td>0.053</td>
<td>0.163</td>
<td>-0.016</td>
<td>0.667</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Phil.</td>
<td>0.777</td>
<td>0.602</td>
<td>0.076</td>
<td>0.020</td>
<td>0.146</td>
<td>0.054</td>
<td>0.048</td>
<td>0.727</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>VF</td>
<td>0.327</td>
<td>0.231</td>
<td>0.086</td>
<td>0.160</td>
<td>0.030</td>
<td>0.185</td>
<td>0.031</td>
<td>0.105</td>
<td>0.871</td>
<td>0.000</td>
</tr>
<tr>
<td>Equi.</td>
<td>0.125</td>
<td>0.110</td>
<td>0.135</td>
<td>0.195</td>
<td>0.010</td>
<td>-0.096</td>
<td>0.065</td>
<td>0.013</td>
<td>0.080</td>
<td>0.947</td>
</tr>
</tbody>
</table>

### Table 3.13: Basic information of the credit-risky portfolio

<table>
<thead>
<tr>
<th>Name</th>
<th>Rating</th>
<th>Coupon</th>
<th>Nominal value</th>
<th>Maturity</th>
<th>Price</th>
<th>pcs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>UniCredit</td>
<td>BBB</td>
<td>1.30%</td>
<td>100 000 €</td>
<td>Jul-24</td>
<td>98.65%</td>
<td>10</td>
</tr>
<tr>
<td>Bayer</td>
<td>BBB+</td>
<td>0.05%</td>
<td>100 000 €</td>
<td>Jun-20</td>
<td>99.08%</td>
<td>10</td>
</tr>
<tr>
<td>Carrefour</td>
<td>BBB</td>
<td>1.25%</td>
<td>100 000 €</td>
<td>Jun-25</td>
<td>102.04%</td>
<td>10</td>
</tr>
<tr>
<td>Peugeot</td>
<td>BBB-</td>
<td>2.38%</td>
<td>50 000 €</td>
<td>Apr-23</td>
<td>105.60%</td>
<td>20</td>
</tr>
<tr>
<td>Danone</td>
<td>BBB+</td>
<td>1.38%</td>
<td>100 000 €</td>
<td>Jun-19</td>
<td>99.83%</td>
<td>10</td>
</tr>
<tr>
<td>Kappa</td>
<td>BB</td>
<td>3.25%</td>
<td>100 000 €</td>
<td>Jun-21</td>
<td>104.69%</td>
<td>10</td>
</tr>
<tr>
<td>Autocam</td>
<td>B+</td>
<td>3.80%</td>
<td>20 000 €</td>
<td>Nov-27</td>
<td>94.35%</td>
<td>50</td>
</tr>
<tr>
<td>Valeo</td>
<td>BBB-</td>
<td>1.63%</td>
<td>100 000 €</td>
<td>Mar-26</td>
<td>100.92%</td>
<td>10</td>
</tr>
<tr>
<td>T-mobile</td>
<td>BB+</td>
<td>6.38%</td>
<td>100 000 €</td>
<td>Mar-25</td>
<td>103.56%</td>
<td>10</td>
</tr>
<tr>
<td>Eni</td>
<td>BBB-</td>
<td>3.75%</td>
<td>100 000 €</td>
<td>Jun-19</td>
<td>100.26%</td>
<td>10</td>
</tr>
</tbody>
</table>

### 3.3.2 Analysis of a credit-risky portfolio

Consider another credit-risky portfolio with ten bonds traded on FSE as well. The total nominal value is 10 million euro and each bond is represented in a nominal value of 1 million euro. The industries of these ten bonds belong to are the same as what we set in the previous high-quality portfolio.
3.3.2.1 Original CreditMetrics model

Table 3.14 presents the correlation matrix of the credit-risky portfolio based on the historical daily stock prices. The stock prices can be found in Appendix A.5. The selected horizon is from October 9th, 2017 to October 8th, 2018. The lowest correlation coefficient is 0.017 between Eni and T-mobile, which is negatively related; while the highest correlation coefficient is between Eni and UniCredit, both of which are Italian companies. Then the Cholesky decomposition matrix based on the correlation matrix is presented in 3.15.

The yield curves from 2018 to 2027 are shown in table 3.8 and figure 3.4 in the previous subchapter. Table 3.16 summarizes the future value of each bond in the credit-risky portfolio according to their rating categories. The highest future value in the portfolio is 209,816 € from Eni, while the lowest one is 23,561 € from Autocam with the worst credit rating.

3.3.2.2 CreditMetrics model based on copulas

The correlation matrix based on copulas is shown in table 3.17. The highest correlation coefficient is 0.467 between Bayer and Valeo; while the lowest correlation
coefficient is 0.003 between Kappa and Autocam. Figure 3.12 presents the numbers of different copulas in the credit-risky portfolio in the form of a pie chart. There are 9 elliptical copulas and 36 Archimedean copulas, most of which are the Frank copula.

Figure 3.13 is the skewness-kurtosis plot of the credit-risky portfolio by using the function `descdist` based on the portfolio value, presented in Appendix A.6. The blue point suggests that the portfolio value is Beta distributed. Besides, the summary statistics illustrates that the distribution is negative skewed and platykurtic. Compared with the high-quality portfolio, the credit-risky portfolio is more volatile because of a higher standard deviation.

```r
# Summary statistics
summary_stats <- descdist(portfolio_value2, discrete = FALSE)

# Display summary statistics
summary_stats
```

**Table 3.14: Correlation matrix of the credit-risky portfolio**

<table>
<thead>
<tr>
<th></th>
<th>UC</th>
<th>Bayer</th>
<th>Carre.</th>
<th>Peug.</th>
<th>Danone</th>
<th>Kappa</th>
<th>AC</th>
<th>Valeo</th>
<th>T.</th>
<th>Eni</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>1.000</td>
<td>0.189</td>
<td>0.205</td>
<td>0.208</td>
<td>0.083</td>
<td>-0.036</td>
<td>0.088</td>
<td>0.175</td>
<td>-0.021</td>
<td>0.467</td>
</tr>
<tr>
<td>Bayer</td>
<td>0.189</td>
<td>1.000</td>
<td>0.224</td>
<td>0.345</td>
<td>0.356</td>
<td>0.201</td>
<td>0.197</td>
<td>0.319</td>
<td>-0.030</td>
<td>0.349</td>
</tr>
<tr>
<td>Carre.</td>
<td>0.205</td>
<td>0.224</td>
<td>1.000</td>
<td>0.066</td>
<td>0.310</td>
<td>0.060</td>
<td>0.028</td>
<td>0.043</td>
<td>0.050</td>
<td>0.165</td>
</tr>
<tr>
<td>Peug.</td>
<td>0.208</td>
<td>0.345</td>
<td>0.066</td>
<td>1.000</td>
<td>0.194</td>
<td>0.134</td>
<td>0.104</td>
<td>0.358</td>
<td>0.033</td>
<td>0.250</td>
</tr>
<tr>
<td>Danone</td>
<td>0.083</td>
<td>0.356</td>
<td>0.310</td>
<td>0.194</td>
<td>1.000</td>
<td>0.157</td>
<td>0.160</td>
<td>0.234</td>
<td>0.055</td>
<td>0.206</td>
</tr>
<tr>
<td>Kappa</td>
<td>-0.036</td>
<td>0.201</td>
<td>0.060</td>
<td>0.134</td>
<td>0.157</td>
<td>1.000</td>
<td>0.025</td>
<td>0.122</td>
<td>0.148</td>
<td>0.060</td>
</tr>
<tr>
<td>AC</td>
<td>0.088</td>
<td>0.197</td>
<td>0.028</td>
<td>0.104</td>
<td>0.160</td>
<td>0.025</td>
<td>1.000</td>
<td>0.148</td>
<td>0.062</td>
<td>0.212</td>
</tr>
<tr>
<td>Valeo</td>
<td>0.175</td>
<td>0.319</td>
<td>0.043</td>
<td>0.358</td>
<td>0.234</td>
<td>0.122</td>
<td>0.148</td>
<td>1.000</td>
<td>0.000</td>
<td>0.223</td>
</tr>
<tr>
<td>T.</td>
<td>-0.021</td>
<td>-0.030</td>
<td>0.050</td>
<td>0.033</td>
<td>0.055</td>
<td>0.148</td>
<td>0.062</td>
<td>0.000</td>
<td>1.000</td>
<td>-0.017</td>
</tr>
<tr>
<td>Eni</td>
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<td>0.349</td>
<td>0.165</td>
<td>0.250</td>
<td>0.206</td>
<td>0.060</td>
<td>0.212</td>
<td>0.223</td>
<td>-0.017</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**Table 3.15: Cholesky matrix of the credit-risky portfolio**

<table>
<thead>
<tr>
<th></th>
<th>UC</th>
<th>Bayer</th>
<th>Carre.</th>
<th>Peug.</th>
<th>Danone</th>
<th>Kappa</th>
<th>AC</th>
<th>Valeo</th>
<th>T.</th>
<th>Eni</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
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<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Peug.</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Danone</td>
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<td>0.237</td>
<td>0.084</td>
<td>0.900</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
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<td>0.000</td>
<td>0.000</td>
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</tr>
<tr>
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<td>0.030</td>
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<td>0.973</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Valeo</td>
<td>0.175</td>
<td>0.291</td>
<td>-0.050</td>
<td>0.247</td>
<td>0.122</td>
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<td>0.060</td>
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</tr>
<tr>
<td>T.</td>
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<td>0.062</td>
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<td>0.052</td>
<td>0.147</td>
<td>0.068</td>
<td>-0.016</td>
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<td>Eni</td>
<td>0.467</td>
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<td>0.077</td>
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<td>0.116</td>
<td>0.034</td>
<td>-0.018</td>
<td>0.828</td>
</tr>
</tbody>
</table>

![Table 3.14: Correlation matrix of the credit-risky portfolio](image)

![Table 3.15: Cholesky matrix of the credit-risky portfolio](image)
Figure 3.12: Numbers of copulas in the credit-risky portfolio

Figure 3.13: Skewness-kurtosis plot
Let’s fit a normal distribution and a Beta distribution by the function `fitdist`. Note that data should be in the range $(0, 1)$ when the Beta distribution is fitted, so we need to scale the data first. Figure 3.14 and figure 3.15 present the plot of the normal and Beta distribution respectively. It is clear that both distributions well fit the portfolio value, but the Beta distribution looks better at tails in the density Q-Q plot. The best fit distribution of the credit-risky portfolio is therefore the Beta distribution.

The Cholesky decomposition matrix of the credit-risky portfolio based on copulas is in table 3.18. The Monte Carlo simulation is conducted by generating the random variables of the Beta distribution instead of the normal distribution. Two required parameters in the Beta distribution can be calculated by $\alpha = \mu \left[ \frac{\mu(1-\mu)}{\sigma^2} - 1 \right]$ and $\beta = \frac{\alpha(1-\mu)}{\mu}$, where $\mu$ and $\sigma$ is the mean and standard deviation of the scaled portfolio value respectively. Then we can convert the same uniformly distributed random variables we used in subchapter 3.3.2.1 into Beta-distributed variables.

The VaR of the portfolio can be computed by the function `cm.CVaR` in the package `CreditMetrics` at different confidence levels as summarized in figure 3.16. The VaR based on copulas are higher than the original VaR at different confidence levels
Chapter 3. Portfolio credit risk model and copulas

Figure 3.14: Plot of the normal distribution

Figure 3.15: Plot of the Beta distribution
Chapter 3. Portfolio credit risk model and copulas

Table 3.17: Correlation matrix of the credit-risky portfolio based on copulas

<table>
<thead>
<tr>
<th></th>
<th>UC</th>
<th>Bayer</th>
<th>Carre.</th>
<th>Peug.</th>
<th>Danone</th>
<th>Kappa</th>
<th>AC</th>
<th>Valeo</th>
<th>T.</th>
<th>Eni</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>1.000</td>
<td>0.202</td>
<td>0.252</td>
<td>0.189</td>
<td>0.091</td>
<td>0.066</td>
<td>0.119</td>
<td>0.198</td>
<td>-0.024</td>
<td>0.432</td>
</tr>
<tr>
<td>Bayer</td>
<td>0.202</td>
<td>1.000</td>
<td>0.286</td>
<td>0.467</td>
<td>0.420</td>
<td>0.175</td>
<td>0.198</td>
<td>0.461</td>
<td>-0.020</td>
<td>0.370</td>
</tr>
<tr>
<td>Carre.</td>
<td>0.252</td>
<td>0.286</td>
<td>1.000</td>
<td>0.141</td>
<td>0.298</td>
<td>0.052</td>
<td>0.085</td>
<td>0.137</td>
<td>0.042</td>
<td>0.190</td>
</tr>
<tr>
<td>Peug.</td>
<td>0.189</td>
<td>0.467</td>
<td>0.141</td>
<td>1.000</td>
<td>0.257</td>
<td>0.198</td>
<td>0.119</td>
<td>0.434</td>
<td>0.056</td>
<td>0.325</td>
</tr>
<tr>
<td>Danone</td>
<td>0.091</td>
<td>0.420</td>
<td>0.298</td>
<td>0.257</td>
<td>1.000</td>
<td>0.152</td>
<td>0.124</td>
<td>0.281</td>
<td>0.070</td>
<td>0.213</td>
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<tr>
<td>Kappa</td>
<td>0.066</td>
<td>0.175</td>
<td>0.052</td>
<td>0.198</td>
<td>0.152</td>
<td>1.000</td>
<td>0.003</td>
<td>0.111</td>
<td>0.180</td>
<td>0.111</td>
</tr>
<tr>
<td>AC</td>
<td>0.119</td>
<td>0.198</td>
<td>0.085</td>
<td>0.119</td>
<td>0.124</td>
<td>0.003</td>
<td>1.000</td>
<td>0.193</td>
<td>0.072</td>
<td>0.192</td>
</tr>
<tr>
<td>Valeo</td>
<td>0.198</td>
<td>0.461</td>
<td>0.137</td>
<td>0.434</td>
<td>0.281</td>
<td>0.111</td>
<td>0.193</td>
<td>1.000</td>
<td>0.008</td>
<td>0.332</td>
</tr>
<tr>
<td>T.</td>
<td>-0.024</td>
<td>-0.020</td>
<td>0.042</td>
<td>0.056</td>
<td>0.070</td>
<td>0.180</td>
<td>0.072</td>
<td>0.008</td>
<td>1.000</td>
<td>-0.007</td>
</tr>
<tr>
<td>Eni</td>
<td>0.432</td>
<td>0.370</td>
<td>0.190</td>
<td>0.325</td>
<td>0.213</td>
<td>0.111</td>
<td>0.192</td>
<td>0.332</td>
<td>-0.007</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 3.18: Cholesky matrix of the credit-risky portfolio based on copulas

<table>
<thead>
<tr>
<th></th>
<th>UC</th>
<th>Bayer</th>
<th>Carre.</th>
<th>Peug.</th>
<th>Danone</th>
<th>Kappa</th>
<th>AC</th>
<th>Valeo</th>
<th>T.</th>
<th>Eni</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Bayer</td>
<td>0.202</td>
<td>0.979</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Carre.</td>
<td>0.252</td>
<td>0.240</td>
<td>0.938</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Peug.</td>
<td>0.189</td>
<td>0.438</td>
<td>-0.013</td>
<td>0.879</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Danone</td>
<td>0.091</td>
<td>0.410</td>
<td>0.188</td>
<td>0.071</td>
<td>0.885</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Kappa</td>
<td>0.066</td>
<td>0.165</td>
<td>-0.005</td>
<td>0.129</td>
<td>0.079</td>
<td>0.972</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>AC</td>
<td>0.119</td>
<td>0.178</td>
<td>0.013</td>
<td>0.022</td>
<td>0.041</td>
<td>-0.041</td>
<td>0.975</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Valeo</td>
<td>0.198</td>
<td>0.430</td>
<td>-0.017</td>
<td>0.237</td>
<td>0.083</td>
<td>-0.010</td>
<td>0.087</td>
<td>0.840</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>T.</td>
<td>-0.024</td>
<td>-0.016</td>
<td>0.055</td>
<td>0.077</td>
<td>0.071</td>
<td>0.174</td>
<td>0.082</td>
<td>-0.011</td>
<td>0.974</td>
<td>0.000</td>
</tr>
<tr>
<td>Eni</td>
<td>0.432</td>
<td>0.289</td>
<td>0.013</td>
<td>0.133</td>
<td>0.049</td>
<td>0.014</td>
<td>0.087</td>
<td>0.095</td>
<td>-0.016</td>
<td>0.832</td>
</tr>
</tbody>
</table>

not only because most of the correlation coefficients in the correlation matrix based on copulas are higher than those in the original correlation matrix, but also because the Monte Carlo simulation is conducted by generating the random variables of the Beta distribution. Moreover, the VaR of two portfolios are summarized in table 3.19, the VaR of the credit-risky portfolio is logically higher than the VaR of the high-quality portfolio at any confidence levels. The absolute difference $\Delta$VaR between the original VaR and the VaR based on copulas are higher with higher confidence levels. The highest $\Delta$VaR is 1 085 € of the credit-risky portfolio when the confidence level is 95%. When we have a look at the corresponding rates of change, all $\Delta$% VaR are less than 1% for the high-quality portfolio, which means the CreditMetrics$^\text{TM}$ model is sufficient enough to measure credit risk associated this portfolio. However, $\Delta$% VaR are higher for the credit-risky portfolio, especially it is equal to 38.46% when the confidence level is 95%. In this case, it is necessary to conduct the CreditMetrics$^\text{TM}$ model based on the copula functions.
Chapter 3. Portfolio credit risk model and copulas

3.4 Summary

This chapter aims at combining the CreditMetrics™ model and the copula functions to better estimate credit risk. We firstly start from the basic description of credit risk models of a single obligor, including the structural and reduced-form models. Then the framework of the well-known CreditMetrics™ model is described in detail, namely credit rating migration, calculation of the present value of a bond, calculation of the discount rate, and credit risk estimation. Credit quality correlation and Monte Carlo simulation are discussed as well.

We employ the CreditMetrics™ model to two different portfolios, one is a high-quality portfolio with ten good-rating bonds, and another is a credit-risky portfolio with ten risky bonds. Bonds in both portfolios are traded in Frankfurt Stock Exchange and the time horizon is from 9th of October, 2017 to 8th of October, 2018. The total nominal value is fixed at 10 million euro and each bond is represented in 1 million euro equally. After that, the CreditMetrics™ model is concerned with the copula functions to recalculate the correlation matrix and most of correlation coefficients are higher than those in the original correlation matrix. Besides, we try to remove the implicit assumption of the normal distribution in the original
CreditMetrics™ model and find the real distribution of the portfolio value to conduct the Monte Carlo simulation using the same uniformly distributed random variables. The real distribution is still the normal distribution for the high-quality portfolio, while it results in the Beta distribution for the credit-risky portfolio. The VaR of the high-quality portfolio are logically lower than the VaR of the credit-risky portfolio no matter at which confidence level, because the riskier the portfolio, the greater value of the VaR. Besides, the original VaR of both portfolios are lower than the VaR calculated based on the copula functions at different confidence levels, which illustrates that the credit risk is underestimated in the original CreditMetrics™ model for both portfolios. Moreover, the difference between the original VaR and the VaR based on the copula functions is more obvious for the credit-risky portfolio, especially when the confidence level is 95%. In my experiment, therefore, it is reasonable to combine the original CreditMetrics™ model and the copula functions for the credit-risky portfolio at any confidence levels, especially when the desired confidence level is 95%. We can also expect that results should probably be similar for more experiments.

### Table 3.19: Summary of VaR of two portfolios

<table>
<thead>
<tr>
<th>Alpha</th>
<th>High-quality portfolio</th>
<th>Credit-risky portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VaR_Original (€)</td>
<td>VaR_Copula (€)</td>
</tr>
<tr>
<td>99.9%</td>
<td>47 273</td>
<td>47 428</td>
</tr>
<tr>
<td>99.5%</td>
<td>18 152</td>
<td>18 235</td>
</tr>
<tr>
<td>99%</td>
<td>9 088</td>
<td>9 149</td>
</tr>
<tr>
<td>95%</td>
<td>432</td>
<td>436</td>
</tr>
</tbody>
</table>

Chapter 3. Portfolio credit risk model and copulas
Chapter 4

Collateralized debt obligation pricing under copula framework

CDO is widely applied to the risk management of banks and other financial institutions as one of most popular credit derivatives in the financial market. Pricing of CDO is a research priority because of not only the rapid development of CDO but also the U.S. subprime credit crisis of 2007 and 2008. Crouhy et al. (2008) and Hull (2009) gives excellent descriptions of the pros and cons of securitization in view of the subprime crisis. Frey and Seydel (2010) technically analyzes the value of securitization as a tool of risk management.

The study of CDO pricing models usually focuses on the probability of defaults of the obligors and the default correlations between the obligors. Copula models are widely extended to CDO pricing and risk measurement. Li (2000) firstly proposes the copula model for portfolio credit risk based on the Gaussian copula. The one-factor Gaussian copula model, the simplest copula model, has already been the benchmark of pricing models in the industry. Schönbucher and Schubert (2001) introduces the general copula models and then Laurent and Gregory (2005) discusses the factor copula models. Under the factor copula framework, Gordy and Jones (2003) analyzes the risks within CDO tranches. The good extensibility of copula models enables them to improve the imitative effects of CDO pricing. Semianalytic approaches for CDO pricing in the factor copula models is developed by Hull and White (2004), Gibson (2004), Andersen and Sidenius (2004), and Laurent and Gregory (2005), among others. Burtschell et al. (2009) compares some popular CDO pricing models related to the bottom-up approach with large portfolio approximation techniques.

In this chapter, we start from the basic structure of CDO, then we introduce the pricing of CDO under copula framework. Since the key issue for pricing CDO is to
determine the cumulative loss distribution function \( F_\infty(x) \), the factor copula models, including the additive factor copulas and Archimedean copulas, are discussed. More specifically, there are seven selected factor copula models, namely Gaussian copula, \( t \) copula, double \( t \) copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings, and Clayton copula. Besides, the large homogeneous portfolio approximation (LHP) by Vasicek (1987) is introduced to derive the approximating distribution of the portfolio loss. There are two main parts in the empirical study. The first part analyzes how the tranche spreads of a CDO will change in consider with different correlations and recovery rates. And then the second part focuses on the ability of different advanced factor copula models to fit the market quotes and correlation skew.

4.1 Structure of CDO

A collateralized debt obligation (CDO) is a structural financial product that pools together credit risky assets, which are called collaterals, and then repackages the asset pool into varying tranches. When the collaterals are bank loans, the CDO is called a collateralized loan obligation (CLO); when they are high-yield bonds, the CDO is called a collateralized bond obligation (CBO). A CDO may also include fixed income securities, subordinated debt, emerging market corporate debt, etc.

The CDO is managed by a sponsoring organization, namely a special purpose vehicle (SPV). The tranches in a CDO are categorized as senior, mezzanine, and junior/subordinated tranches according to the level of credit quality, which is usually a credit rating received from a rating agency (Moody’s, Fitch, S&P, etc.). The standard prioritization scheme is a simple subordination: senior CDO notes are paid before mezzanine and lower subordinated notes are paid, with any residual cash flows paid to junior notes. This mechanism is known as the waterfall. Figure 4.1 shows a simplified CDO structure. The junior tranche is usually called the equity tranche because the holders of the junior tranche are similar to the equity holders in a corporation in receiving the residual cash flows.

A tranche is typically a certain loss range defined by attachment and detachment points. The attachment point is the lower bound of the range and the detachment point is the upper bound. Table 4.1 summarizes the attachment and detachment points, \( K^L \) and \( K^U \), of different tranches in both the US and the European market.
Figure 4.1: Simplified CDO structure
Source: McNeil et al. (2015, pp.477)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>CDX US</th>
<th>iTraxx Europe</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(K_i^L(%))</td>
<td>(K_i^U(%))</td>
</tr>
<tr>
<td>Equity</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Mezzanine junior</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>Mezzanine senior</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Senior junior</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Senior</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>Super senior</td>
<td>30</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 4.1: Attachment and detachment points in US and Europe
Source: McNeil et al. (2015, pp.483)

Figure 4.2 presents a so-called correlation smile. It reveals that the mezzanine tranches are typically associated with a lower compound correlation than equity or senior tranches. Besides, the senior tranche shows a higher correlation than the equity tranche. Market data implies that Gaussian copula, which provides a flat implied correlation, fails to model fat tails and underestimate the probability of observing the extreme values.

### 4.2 Pricing of CDO under copula framework

CDO can be priced using the copula framework following the representation given by Laurent and Gregory (2005). Consider a CDO of \(n\) underlying assets with default times \(\tau_i\), loss given default \(\delta_i\), and notional amount \(A_i\). The marginal probability distribution functions of the underlying assets at the time of default are \(F_1(t_1), F_2(t_2), \ldots, F_n(t_n)\). The joint distribution function is \(F\). There exists a copula
Chapter 4. Collateralized debt obligation pricing under copula framework

Figure 4.2: Implied correlation smile for iTraxx tranches
Source: Burtschell et al. (2009)

\[ C : [0, 1]^n \rightarrow [0, 1] \text{ such that:} \]

\[ F(t_1, t_2, \ldots, t_n) = C(F_1(t_1), F_2(t_2), \ldots, F_n(t_n)). \]  

(4.1)

Generate \( n \) uniformly distributed random variables \( U_i \) based on the selected copula first and then transform \( U_i \) into default times \( \tau_i \) in ascending order. Compute the default intensity \( \lambda(t) \) and default times \( \tau_i \) is given by \( \tau_i = -\frac{\ln U_i}{\lambda} \). The cumulative loss of the portfolio up to time \( t \) is given by

\[ L(t) = \sum_{i=1}^{n} \delta_i A_i \mathbf{1}_{\{\tau_i \leq t\}}, \forall i = 1, 2, \ldots, n. \]  

(4.2)

Assume that there is a tranche \( \gamma \) and the attachment/detachment points are denoted by \( (K_L^\gamma, K_U^\gamma) \), then the cumulative loss of the tranche is

\[ L(K_L^\gamma, K_U^\gamma, t) = \begin{cases} 
0, & \text{if } L(t) < K_L^\gamma \\
L(t) < K_L^\gamma, & \text{if } K_L^\gamma \leq L(t) \leq K_U^\gamma \\
K_U^\gamma - K_L^\gamma, & \text{if } L(t) > K_U^\gamma.
\end{cases} \]  

(4.3)
Chapter 4. Collateralized debt obligation pricing under copula framework

It can be summarized as the payoff a call spread:

\[
L(K_L^γ, K_U^γ, t) = \max \{ \min[L(t), K_U^γ] - K_L^γ, 0 \} = \{ \min[L(t), K_U^γ] - K_L^γ, 0 \}^+.
\] (4.4)

The expected value of the cumulative tranche loss with the continuous portfolio loss distribution function \( F_∞(x) \) is

\[
E[L(K_L^γ, K_U^γ, t)] = \frac{1}{K_U^γ - K_L^γ} \int_{K_L^γ}^{K_U^γ} [\min(x, K_U^γ) - K_L^γ] dF_∞(x)
\]

\[
= \frac{1}{K_U^γ - K_L^γ} \left[ \int_{K_L^γ}^{1} (x - K_L^γ) dF_∞(x) - \int_{K_L^γ}^{1} (x - K_U^γ) dF_∞(x) \right].
\] (4.5)

The expected value of the default leg (a payoff when the default occurs) and the premium leg (periodic payments) can be computed respectively by

\[
E(DL) = E[\int_0^T B(0, t) dL(K_L^γ, K_U^γ, t)],
\] (4.6)

\[
E(PL) = E[\sum_{i=1}^n s \Delta t_i B(0, t) \min\{ \max[K_U^γ - L(t_i), 0], K_U^γ - K_L^γ \}],
\] (4.7)

where \( T = t_n \) is the maturity, \( B(0, t) \) is the discount factor until time \( t \), \( s \) is the par spread of the tranche, \( \Delta t_i = t_i - t_{i-1} \), and \( L(t_i) = \delta_i A_i 1_{\{ \gamma \leq t_i \}} \).

Based on the general semi-analytic approach, each tranche conducts a premium so that the premium leg equals the default leg, namely \( E(PL) = E(DL) \). The par spread \( s^* \) is therefore

\[
s^* = \frac{E[\int_0^T B(0, t) dL(K_L^γ, K_U^γ, t)]}{E[\sum_{i=1}^n \Delta t_i B(0, t) \min\{ \max[K_U^γ - L(t_i), 0], K_U^γ - K_L^γ \}]}.
\] (4.8)

It is clear that the key issue for pricing CDO is to determine the cumulative loss distribution function \( F_∞(x) \), which is the important element of the expected value of the cumulative tranche loss \( E[L(K_L^γ, K_U^γ, t)] \). However, it is not easy to derive \( F_∞(x) \) because of influences of the default correlation between the reference entities. Thus, the remaining introduce factor copula model to derive the portfolio cumulative loss distribution \( F_∞(x) \).
Chapter 4. Collateralized debt obligation pricing under copula framework

4.3 Factor copula model

One factor copula model is a copula model with the latent variables decomposed into one systematic or common factor and \( n \) idiosyncratic factors. It assumes that those factors are distributed based on a certain copula function. The value of the \( i \)-th asset is \( V_i = f(Y, Z_i) \), \( \forall i = 1, 2, \cdots, n \), where \( Y \) is the systematic factor, \( Z_i \) are the idiosyncratic factors. \( Y \) and \( Z_i \) are mutually independent random variables. The first factor copula model for portfolio credit risk was given by Li (2000), which is based on the Gaussian copula. General copula models were introduced by Schönbucher and Schubert (2001).

**Additive factor copula** is widely used in CDO pricing. The function \( V_i = f(Y, Z_i) \) means the systematic factor \( Y \) and the idiosyncratic factors \( Z_i \) are additive. Therefore, the value of the \( i \)-th asset is

\[
V_i = \rho_i Y + \sqrt{1 - \rho_i^2} Z_i, \forall i = 1, 2, \cdots, n,
\]

where \( \rho_i \in [0, 1] \) is the correlation coefficient between the \( i \)-th asset and the systematic factor. Denote the probability distribution function of \( Y, Z_i, V_i \) by \( F_Y, F_{Z_i}, F_{V_i} \) respectively.

Let \( K_i \) be the default barrier of the \( i \)-th asset, the default time is defined as

\[
\tau_i = \inf\{t \geq 0 : V_i \leq K_i\}, \forall i = 1, 2, \cdots, n.
\]

The default barrier can be given by \( K_i = F_{V_i}^{-1}(Q_i(t)) \) and \( Q_i(t) \) is the \( i \)-th default probability for the time \( t \).

In the general case, the conditional default probability of the \( i \)-th asset is

\[
p_i(y) = P(z_i \leq \frac{K_i - \rho_i y}{\sqrt{1 - \rho_i^2}} | Y = y) = F_{Z_i}[\frac{F_{V_i}^{-1}(Q_i(t)) - \rho_i y}{\sqrt{1 - \rho_i^2}}].
\]

**Archimedean copulas** usually have the explicit forms which makes them good at modelling portfolio credit risk. Marshall and Olkin (1988) introduce Archimedean copulas for the first time. Famous Archimedean copulas include Gumbel, Clayton, and Frank copula.

Consider a positive random variable \( Y \) with probability density \( f(x) \) and denote the Laplace transform of \( f \) by \( \psi \). We have \( \psi(s) = \int_0^\infty f(x)e^{-sx} \, dx \). Define the value of
the $i$-th asset as
\begin{equation}
V_i = \psi\left(-\frac{\ln Z_i}{Y}\right), \forall i = 1, 2, \cdots, n.
\end{equation}

Then the conditional default probability of the $i$-th asset is
\begin{equation}
p_i(y) = P(V_i \leq K_i \mid Y = y) = \exp\{-y\psi^{-1}[F_{V_i}^{-1}(Q_i(t))]\}.
\end{equation}

Laurent and Gregory (2005) and Friend and Rogge (2005) discuss Clayton copula in CDO pricing and risk measurement. Besides, Schlogel and O’Kane (2005) and Rogge and Schönbucher (2003) introduce other Archimedean copulas, such as Gumbel and Frank copula, in modelling credit risk. Table 4.2 summarizes specifications for the distribution and the corresponding Laplace transforms.

<table>
<thead>
<tr>
<th>Copula</th>
<th>Gumbel copula</th>
<th>Clayton copula</th>
<th>Frank copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi^{-1}(s)$</td>
<td>$(- \ln t)^\theta$</td>
<td>$s^{-\theta} - 1$</td>
<td>$- \ln \frac{e^{s \frac{\theta - 1}{\theta}} - 1}{e^{s \theta} - 1}$</td>
</tr>
<tr>
<td>$\psi(s)$</td>
<td>$\exp(-s^{\frac{\theta}{\theta - 1}})$</td>
<td>$(1 + s)^{-\frac{1}{\theta}}$</td>
<td>$-\frac{1}{\theta}\ln \left[1 - e^{-s(1 - e^{-\theta})}\right]$</td>
</tr>
<tr>
<td>Parameter</td>
<td>$\theta &gt; 1$</td>
<td>$\theta &gt; 0$</td>
<td>$\theta \in \mathbb{R} \setminus {0}$</td>
</tr>
<tr>
<td>Distribution</td>
<td>$\alpha$-stable, $\alpha = \frac{1}{\theta}$</td>
<td>Gamma $(\frac{1}{\theta})$</td>
<td>Logarithmic series on $\mathbb{N}_+$ with $\alpha = 1 - e^{-\theta}$</td>
</tr>
<tr>
<td>Density</td>
<td>(no closed-form is known)</td>
<td>$\frac{1}{\Gamma(\frac{1}{\theta})}e^{-x}x^{\frac{1-\alpha}{\theta}}$</td>
<td>$P[Y = k] = \frac{-1}{\ln(\alpha - 1)} \frac{\alpha^k}{k}$</td>
</tr>
</tbody>
</table>

Table 4.2: Basic information of Archimedean copulas

### 4.3.1 Gaussian copula

The most frequently used one is one-factor Gaussian copula model, first proposed by Vasicek (2002). The systematic factor $Y$ and the idiosyncratic factors $Z_i$ are independent and identically distributed standard normal random variables, then $V_i$ are also normally distributed. Then the conditional default probability of the $i$-th asset can be specified as
\begin{equation}
p_i(y) = \Phi\left[\frac{\Phi^{-1}(Q_i(t)) - \rho_i y}{\sqrt{1 - \rho_i^2}}\right].
\end{equation}

When $\rho = 0$, default times $\tau_i$ are mutually independent; when $\rho = 1$, $V_i = Y$ and $p_i(y) = 1_{\{y \leq \phi^{-1}(F_i(t))\}}$. 

4.3.2 \( t \) copula

\( t \) copula, a simple extension of Gaussian copula, is considered to portfolio credit risk by Marshall et al. (2003), Andersen et al. (2003), Frey and McNeil (2003), Demarta and McNeil (2005), and Schlogel and O’Kane (2005) because \( t \) copula is fat tailed.

In \( t \) copula, \( V_i \) follow a \( t \) distribution with \( \nu \) degrees of freedom. We consider the symmetric situation where \( V_i = \sqrt{W} X_i = \rho_i Y + \sqrt{1 - \rho_i^2} Z_i \). Then the value of \( i \)-th asset is

\[
V_i = \rho_i \cdot \sqrt{W} \cdot Y + \sqrt{1 - \rho_i^2} \cdot \sqrt{W} \cdot Z_i = \sqrt{W} (\rho_i Y + \sqrt{1 - \rho_i^2} Z_i).
\] (4.15)

Note that \( Y \) and \( Z_i \) are independent Gaussian random variables. \( W \) is independent of \( X_i \) and has an inverse Gamma distribution with the parameter \( \frac{\nu}{2} \), namely \( W \sim \Gamma^{-1}(\frac{\nu}{2}, \frac{\nu}{2}) \), or equivalently \( \frac{\nu}{W} \sim \chi^2_{\nu} \).

The conditional default probability of the \( i \)-th asset under \( t \) copula is

\[
p_i(y) = \Phi\left[\frac{t^{-1}_\nu(Q_i(t)) - \rho_i \sqrt{W} y}{\sqrt{1 - \rho_i^2} \cdot \sqrt{W}}\right] = \Phi\left[\frac{\frac{1}{\sqrt{W}} \cdot t^{-1}_\nu(Q_i(t)) - \rho_i y}{\sqrt{1 - \rho_i^2}}\right].
\] (4.16)

4.3.3 Double \( t \) copula

Double \( t \) copula applied in pricing of CDOs was proposed by Hull and White (2004). Further discussion can be found in Cousin and Laurent (2007). In double \( t \) copula, \( Y \) and \( Z_i \) are independent random variables and follow univariate \( t \) distributions with \( \nu \) and \( \bar{\nu} \) degrees of freedom, the value of the \( i \)-th asset is

\[
V_i = \rho_i \cdot \sqrt{\frac{\nu - 2}{\nu}} \cdot Y + \sqrt{1 - \rho_i^2} \cdot \sqrt{\frac{\bar{\nu} - 2}{\bar{\nu}}} \cdot Z_i.
\] (4.17)

Because \( t \) distributions cannot be easily additive, the distribution of \( V_i \) is not \( t \) distribution and computed based on \( \rho \). The conditional default probability of the \( i \)-th asset under double \( t \) copula is

\[
p_i(y) = t_{\bar{\nu}}\left[\frac{F^{-1}_\nu(Q_i(t)) - \rho_i \cdot \sqrt{\frac{\nu - 2}{\nu}} y}{\sqrt{1 - \rho_i^2} \cdot \sqrt{\frac{\nu - 2}{\nu}}}\right]
\] (4.18)

\[
= t_{\bar{\nu}}\left[\frac{\nu - 2}{\bar{\nu}} \cdot F^{-1}_\nu(Q_i(t)) - \rho_i \cdot \sqrt{\frac{\nu - 2}{\nu}} y}{\sqrt{1 - \rho_i^2}}\right].
\]
4.3.4 NIG copula

Normal inverse Gaussian (NIG) distribution is a mixture of normal and inverse Gaussian distributions and is a special case of the generalized hyperbolic distribution by Barndorff-Nielsen. Kalemanova et al. (2005) proved that NIG distribution fits the market data well. Basic definition and properties of the NIG distribution can be found in Appendix B.1.

The value of the \( i \)-th asset in NIG copula looks identical to Gaussian representation:

\[
V_i = \rho_i Y + \sqrt{1 - \rho_i^2} Z_i, \tag{4.19}
\]

where \( Y \) and \( Z_i \) are independent normal inverse Gaussian random variables, satisfying \( Y \sim NIG(\alpha, \beta, \frac{-\alpha \beta}{\sqrt{\alpha^2 - \beta^2}}, \alpha) \) and \( Z_i \sim NIG\left(\frac{\sqrt{1 - \rho_i^2}}{\rho_i} \alpha, \frac{\sqrt{1 - \rho_i^2}}{\rho_i} \beta, -\frac{\alpha \beta}{\sqrt{\alpha^2 - \beta^2}}, \frac{\sqrt{1 - \rho_i^2}}{\rho_i} \alpha \right) \) with parameters \( \alpha > 0 \) and \( \beta > 0 \). Then \( V_i \) are also normal inverse Gaussian distributed, namely \( V_i \sim NIG\left(\frac{\alpha \rho_i}{\rho_i}, \frac{\beta \rho_i}{\rho_i}, -\frac{1}{\rho_i} \frac{\alpha \beta}{\sqrt{\alpha^2 - \beta^2}}, \frac{\alpha \rho_i}{\rho_i} \right) \).

Denote the probability distribution function by \( F_{NIG}(x; \alpha, \beta, \frac{-\alpha \beta}{\sqrt{\alpha^2 - \beta^2}}, \alpha) \) for simplification. Similarly, the probability distribution functions of \( Y \) and \( Z_i \) are \( F_{NIG(1)}(x) \) and \( F_{NIG(\sqrt{1 - \rho_i^2})}(x) \). The conditional default probability of the \( i \)-th asset under NIG copula is

\[
p_i(y) = F_{NIG(\sqrt{1 - \rho_i^2})}^{-1}\left(\frac{F_{NIG(\frac{1}{\rho_i})}(Q_i(t)) - \rho_i y}{\sqrt{1 - \rho_i^2}}\right), \tag{4.20}\]

4.3.5 Gaussian copula with stochastic correlation

Stochastic correlation model is also an extension of Gaussian copula introduced by Burtschell et al. (2007). In stochastic correlation model, the \( i \)-th asset value is

\[
V_i = \tilde{\rho}_i Y + \sqrt{1 - \tilde{\rho}_i^2} Z_i, \tag{4.21}\]

where \( \tilde{\rho}_i \) are random variables and \( \tilde{\rho}_i \in [0, 1] \) and independent from \( Y \) and \( Z_i \). \( Y \) and \( Z_i \) are Gaussian random variables, so \( V_i \) follow Gaussian distribution. Assume that \( \tilde{\rho}_i \) can take only two values, \( \rho \) and \( \eta \), with probabilities \( p \) and \( 1 - p \):

\[
\tilde{\rho}_i = B_i \rho + (1 - B_i) \eta, \tag{4.22}\]
where $B_i$ are Bernoulli random variables and also independent from $Y$ and $Z_i$. Thus, the value of the $i$-th asset can be rewritten as

$$V_i = [B_i \rho + (1 - B_i) \eta]Y + \sqrt{1 - [B_i \rho + (1 - B_i) \eta]^2}Z_i.$$  

(4.23)

The conditional default probability of the $i$-th asset under stochastic correlation model is

$$p_i(y) = p \Phi(\Phi^{-1}(Q_i(t)) - \rho y \sqrt{1 - \rho^2}) + (1 - p) \Phi(\Phi^{-1}(Q_i(t)) - \eta y \sqrt{1 - \eta^2}).$$  

(4.24)

### 4.3.6 Gaussian copula with random factor loadings

Random factor loading (RFL) model was introduced by Anderson and Sidenius (2004) to better fit the correlation skew observed. The main idea of RFL model is to make factor loadings a function of the systematic factors. The value of the $i$-th asset can be given by

$$V_i = \rho_{ij}(Y_j)Y + v_i Z_i + m_i, \forall i = 1, 2, \cdots, nand j = 1, 2, \cdots, d,$$

(4.25)

where $Y$ and $Z_i$ are independent standard normal random variables, $Y_j$ is a random variable with $d$-dimension and $\rho_{ij}$ are factor loadings with a two-point distribution:

$$\rho_{ij}(Y_j) = \begin{cases} 
\alpha_{ij}, & Y_j \leq \theta_{ij} \\
\beta_{ij}, & Y_j > \theta_{ij} 
\end{cases}$$

(4.26)

where $\alpha_{ij}$, $\beta_{ij}$ are positive constants and $\theta_{ij} \in \mathbb{R}$. Define that loading $j$ takes $\alpha_{ij}$ with probability $\Phi(\theta_{ij})$ and $\beta_{ij}$ with probability $1 - \Phi(\theta_{ij})$. Parameters $m_i$ and $v_i$ are chosen to ensure that $V_i$ has zero mean and unit variance distribution:

$$m_i := -E[\rho_i(Y)Y] = -\sum_{j=1}^{d} (-\alpha_{ij} \phi(\theta_{ij}) + \beta_{ij} \phi(\theta_{ij})), $$

(4.27)

$$v_i := \sqrt{1 - \text{Var}[\rho_i(Y)Y]} = \sqrt{1 - \sum_{j=1}^{d} \text{Var}[\rho_{ij}(Y_j)Y_j]},$$

(4.28)

where $\text{Var}[\rho_{ij}(Y_j)Y_j] = \alpha_{ij}^2(\Phi(\theta_{ij}) - \theta_{ij} \phi(\theta_{ij})) + \beta_{ij}^2(\theta_{ij} \phi(\theta_{ij}) + 1 - \Phi(\theta_{ij})) - (-\alpha_{ij} \phi(\theta_{ij}) + \beta_{ij} \phi(\theta_{ij}))^2$. The proof of the deviation of $m_i$ and $v_i$ can be found in Appendix B.2.
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Assume the dimension of $Y$ to be 1, then the conditional default probability of the $i$-th asset under Gaussian copula with random factor loadings is

\[
p_i(y) = \Phi\left[\frac{\Phi^{-1}(Q_i(t)) - \rho_i(y_i)y - m_i}{v_i}\right] = \Phi\left[\frac{\Phi^{-1}(Q_i(t)) - (\alpha_i 1_{y<\theta}y + \beta_i 1_{y>\theta}y) - m_i}{v_i}\right].
\] (4.29)

### 4.3.7 Clayton copula

The typical example of Archimedean copula for CDO pricing is Clayton copula mentioned in Schönbucher and Schubert (2001), Schönbucher (2002), Gregory and Laurent (2003), Rogge and Schönbucher (2003), Madan et al. (2004), Laurent and Gregory (2005), Schloegl and O’Kane (2005), and Friend and Rogge (2005). Consider a positive random variable follows a standard Gamma distribution with the parameter $\frac{1}{\theta}$, where $\theta > 0$. The probability density is given by

\[
f(x) = \frac{1}{\Gamma(\frac{1}{\theta})} e^{-x \frac{1}{\theta^2}}, x > 0.
\] (4.30)

The Laplace transform of $f$ is

\[
\phi(s) = \int_0^\infty f(x)e^{-sx}dx = (1 + s)^{-\frac{1}{\theta}}.
\] (4.31)

The conditional default probability of the $i$-th asset under Clayton copula is

\[
p_i(y) = \exp \left[ y(1 - Q_i(t)^{-\theta}) \right].
\] (4.32)

### 4.4 Loss distribution of the large homogeneous portfolio

There exists a so-called large homogeneous portfolio approximation (LHP) that allows to derive an analytical situation for the portfolio loss distribution and the then the expected value of the cumulative tranche loss. The approximation, proposed by Vasicek (1987), assumes that the number of the obligors $n$ in the portfolio is extremely large. All obligors are homogeneous, which means they are identical in notional amounts, recovery rates, and unconditional default probabilities. Thus, for
additive factor copulas, the approximating distribution of the portfolio loss is

\[
F_\infty(x) = P[p_i(y) \leq x] = P\{F_Z[F^{-1}_V(Q_i(t)) - \rho_i y] \leq x\} \\
= P[y \geq \frac{F^{-1}_V(Q_i(t)) - \sqrt{1 - \rho_i^2 F^{-1}_Z(x)}}{\rho_i}] \\
= 1 - F_Y[\frac{F^{-1}_V(Q_i(t)) - \sqrt{1 - \rho_i^2 F^{-1}_Z(x)}}{\rho_i}].
\]

(4.33)

Similarly, for Archimedean copula models, the approximating distribution of the portfolio loss is

\[
F_\infty(x) = P[p_i(y) \leq x] = P[\exp\{-y\psi^{-1}[F^{-1}_V(Q_i(t))}\} \leq x] \\
= P[y \geq -\ln \left(\frac{y\psi^{-1}[F^{-1}_V(Q_i(t))]}{\psi^{-1}[F^{-1}_V(Q_i(t))]}\right)] \\
= 1 - F_Y[-\frac{\ln (x)}{\psi^{-1}[F^{-1}_V(Q_i(t))]}].
\]

(4.34)

Therefore, the approximating distributions of the portfolio loss of discussed copula functions in the previous sub-chapter can be summarized as follows:

1. Gaussian copula:

\[
F_\infty(x) = \Phi\left[\frac{\sqrt{1 - \rho_i^2} \Phi^{-1}(x) - \Phi^{-1}(Q_i(t))}{\rho_i}\right].
\]

(4.35)

2. \(t\) copula:

\[
F_\infty(x) = \Phi\left[\frac{\sqrt{1 - \rho_i^2} \Phi^{-1}(x) - \frac{1}{\sqrt{\nu}} t^{-1}(Q_i(t))}{\rho_i}\right].
\]

(4.36)

3. Double \(t\) copula:

\[
F_\infty(x) = 1 - t_{\nu}\left[\frac{\nu - 2}{\nu} \cdot \frac{F^{-1}_V(Q_i(t)) - \sqrt{1 - \rho_i^2} \cdot \frac{\nu - 2}{\nu} t^{-1}(x)}{\rho_i}\right].
\]

(4.37)

4. NIG copula:

\[
F_\infty(x) = F_{\text{NIG}(1)}\left[\frac{\sqrt{1 - \rho_i^2} F^{-1}_{\text{NIG}(\sqrt{1 - \rho_i^2})}(x) - F^{-1}_{\text{NIG}(\frac{1}{\rho_i})}(Q_i(t))}{\rho_i}\right].
\]

(4.38)
5. Gaussian copula with stochastic correlation:

\[
F_\infty(x) = p \Phi\left(\sqrt{1 - \rho^2} \Phi^{-1}(x) - \Phi^{-1}(Q_i(t))\right) \frac{\rho}{\rho} + (1 - p) \Phi\left(\sqrt{1 - \eta^2} \Phi^{-1}(x) - \Phi^{-1}(Q_i(t))\right) \frac{\eta}{\eta}.
\] (4.39)

6. Gaussian copula with random factor loadings:

\[
F_\infty(x) = \Phi\left[\min\left(\frac{\Omega(x)}{\alpha_i}, \theta_i\right)\right] + 1_{\Omega(x) > \theta_i}\left[\Phi\left(\frac{\Omega(x)}{\beta_i}\right) - \Phi(\theta_i)\right].
\] (4.40)

where \(\Omega(x) := \Phi^{-1}(Q_i(t)) - v_i \Phi^{-1}(x) - m_i\).

7. Clayton copula:

\[
F_\infty(x) = F_Y\left[\frac{\ln(x)}{1 - Q_i(t)^{-\theta}}\right].
\] (4.41)

### 4.5 Empirical study

The par spread of a CDO is determined by several factors, including the correlations, recovery rates, maturities, copulas, and so on. In this subchapter, we will conduct a sensitivity analysis in order to have a look at how the tranche spreads of a CDO will change according to different correlations and recovery rates. Both the correlations and recovery rates are set from 0 to 0.8 with the increment of 0.2. The default time is generated by the multinomial Gaussian copula with different correlation matrices. The calculation of the present values of default leg and premium leg follows the equations (4.6) and (4.7). Then the tranche spreads for each tranche can be summarized in a matrix with 5 rows and 5 columns for convenient to analyze the effects of the different correlations and recovery rates on the tranche spreads of a CDO.

Besides, we will fit different advanced copula models described in the previous subchapter to price the CDO based on the market quotes, including Gaussian copula, double t copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings, and Clayton copula. The Dow Jones iTraxx Europe tranches with 5-year maturity is considered from 5th of January 2015 to 10th of May 2016. The recovery rate is 40% and the the risk-free interest rate is 5%. The parameters of six copula models are calibrated to fit the equity tranche by the simplex downhill solver and the gradient method. Six copula models are analyzed...
and compared in terms of the absolute errors, which is the difference between the model implied spreads and the market spreads across all tranches.

### 4.5.1 CDO pricing with correlations and recovery rates

Consider a synthetic CDO composed by 10 homogeneous reference entities with the same notional amount of 100 €. The maturity is 5 years and the risk-free interest rate is 0.05. The constant single name default swap curve is set at 150 basis points. The number of simulations is 100 000. We can insert these basic parameters in R first.

```r
# Basic parameters
ref_ent <- 10
capital <- 100
Amount <- matrix(100,ref_ent,1)
T <- 5
r <- 0.05
spread <- 150/10000
N <- 100000
```

In order to have a look at the relationship between CDO pricing and different recovery rates, recovery rate ranges from 0 to 0.8 with the increment of 0.2. Because the term structure of the CDS is flat, hazard rate = spread/(1 − recovery rate). The corresponding hazard rates are presented in table 4.3.

```r
# Recovery rate ranges from 0 to 0.8
RR <- matrix(0,ref_ent,5)
hazard <- matrix(0,5,1)
for (RR_cycle in 1:5) {
  RR[,RR_cycle] <- (0.2*RR_cycle)-0.2
  hazard[RR_cycle] <- spread/(1-RR[1,RR_cycle])
}
```

<table>
<thead>
<tr>
<th>Recovery rate</th>
<th>Hazard rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.01500</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01875</td>
</tr>
<tr>
<td>0.4</td>
<td>0.02500</td>
</tr>
<tr>
<td>0.6</td>
<td>0.03750</td>
</tr>
<tr>
<td>0.8</td>
<td>0.07500</td>
</tr>
</tbody>
</table>

**Table 4.3:** Recovery rates and corresponding hazard rates

Moreover, the constant pairwise correlation ranges from 0 to 0.8 with the increment of 0.2. The correlation coefficient in diagonal line should always be 1.

```r
# Constant pairwise correlation ranges from 0 to 0.8
R <- seq(0,0.8,by=0.2)
start_matrix <- matrix(0,ref_ent,ref_ent)
R_matrix <- vector('list',5)
for (R_cycle in 1:5) {
  R_matrix[[R_cycle]] <- start_matrix
  R_matrix[[R_cycle]][,1] <- R_cycle
  R_matrix[[R_cycle]][1,] <- R
  for (i in 2:ref_ent) {
    R_matrix[[R_cycle]][i,i] <- 1
    for (j in 2:i) {
      R_matrix[[R_cycle]][i,j] <- R_matrix[[R_cycle]][j,i]
    }
  }
}
```
The function of default time is defined with different correlations and recovery rates. Pseudo times are generated by the multinomial Gaussian copula with different correlation matrices. By dividing the sorted pseudo times by the corresponding hazard rates, we can generate the vector of default time.

\[
\text{def_time} \leftarrow \text{gaussian_time}(\text{R_matrix}[\text{R_cycle}], \text{hazard}[\text{R_cycle}])
\]

The attachment and detachment points are fixed for three tranches as presented in Table 4.4, namely 0%-3%, 3%-14%, and 14%-100%.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Attachment</th>
<th>Detachment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Mezzanine</td>
<td>30</td>
<td>140</td>
</tr>
<tr>
<td>Senior</td>
<td>140</td>
<td>1000</td>
</tr>
</tbody>
</table>

**Table 4.4:** Attachment and detachment points

In order to simplify the calculation of tranche spreads, we can define the function of the present values of default leg and premium leg including several loops and if statements according to (4.6) and (4.7).
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```r
if (total_loss<A) {PV_def <- 0}
if (total_loss>A & total_loss<D) {
  for (i in 1:ref_ent) {
    if (def_time[1,i]<T) {
      indicator <- loss[i]
      if (indicator>A) {
        if (c==0) {
          disc_fact_def <- (1+r)^(-def_time[1,i])
          PV_def <- (indicator-A)*disc_fact_def
          c <- 1
        } else {
          disc_fact_def <- (1+r)^(-def_time[1,i])
          PV_def <- loss[i]*disc_fact_def
        }
      } else {
        disc_fact_def <- (1+r)^(-def_time[1,i])
        PV_def <- loss[i]*disc_fact_def
      }
    }
  }
}
if (total_loss>D) {
  for (i in 1:ref_ent) {
    if (def_time[1,i]<T) {
      indicator <- loss[i]
      if (indicator>A & indicator<D) {
        if (c==0) {
          disc_fact_def <- (1+r)^(-def_time[1,i])
          PV_def <- (indicator-A)*disc_fact_def
          c <- 1
        } else {
          disc_fact_def <- (1+r)^(-def_time[1,i])
          PV_def <- loss[i]*disc_fact_def
        }
      } else {
        disc_fact_def <- (1+r)^(-def_time[1,i])
        PV_def <- (indicator-D)*disc_fact_def
        c <- 2
      }
    }
  }
}
for (i in 1:T) {
  for (j in 1:ref_ent) {
    if (def_time[1,j]<i) {
      periodic_loss[i] <- (1-RR)*capital
    }
  }
  out_capital[i] <- pmin(pmax(D-periodic_loss[i],0),D-A)
  fee[i] <- ((1+r)^(-i))*out_capital[i]
  PV_premium <- fee[i]
}
return(list(PV_def,PV_premium))
```

Now we can compute the tranche spreads for equity, mezzanine, and senior tranche by calls to the functions of default time and cash flows according to (4.8). Each tranche spread is initialized by a vector of zeros with 5 rows and 5 columns because there are 5 recovery rates and 5 correlation coefficients. Table 4.5, table 4.6, and table 4.7 continuously summarize the tranche spreads for equity, mezzanine, and senior tranche.

# Spreads for each tranche
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```r
price_eq <- matrix(0,5,5)
price_mezz <- matrix(0,5,5)
price_sen <- matrix(0,5,5)
S_fees <- matrix(0,5,3)
S_default <- matrix(0,5,3)
M_fees <- matrix(0,5,3)
M_default <- matrix(0,5,3)
for (n in 1:N) {
    for (RR_cycle in 1:5) {
        for (R_cycle in 1:5) {
            PV_def <- cash_flow(T,def_time[n,R,R_cycle],r,capital,A[u],D[u])[[1]]
            PV_premium <- cash_flow(T,def_time[n,R,R_cycle],r,capital,A[u],D[u])[[2]]
            S_fees[R,R_cycle,u] <- PV_premium
            S_default[R,R_cycle,u] <- PV_def
            M_fees[R,R_cycle,u] <- S_fees[R,R_cycle,u]/N
            M_default[R,R_cycle,u] <- S_default[R,R_cycle,u]/N
            price_eq <- (M_default[R,R_cycle,1]/M_fees[R,R_cycle,1])*10000
            price_mezz <- (M_default[R,R_cycle,2]/M_fees[R,R_cycle,2])*10000
            price_sen <- (M_default[R,R_cycle,3]/M_fees[R,R_cycle,3])*10000
        }
    }
}
```

Moreover, we can present how the tranche spreads change based on different correlations and recovery rates in a visualized way of 3D surface plots as shown in figure 4.3, figure 4.4, and figure 4.5 respectively.

```r
#Surface plots
Equity_tranche <- persp(seq(0,0.8,by=0.2),R,price_eq,col='lightskyblue',theta=50,phi=30,expand=0.6,ticktype='detailed', shade=0.3,ltheta=120,xlab='Correlation',ylab='Recovery rate',zlab='Tranche spread (bps per annum)')
Mezzanine_tranche <- persp(seq(0,0.8,by=0.2),R,price_mezz,col='lightskyblue',theta=50,phi=30,expand=0.6,ticktype='detailed', shade=0.3,ltheta=120,xlab='Correlation',ylab='Recovery rate',zlab='Tranche spread (bps per annum)')
```

### Table 4.5: Equity tranche spreads (bps per annum)

<table>
<thead>
<tr>
<th>RR</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2330</td>
<td>2020</td>
<td>2000</td>
<td>1872</td>
<td>1509</td>
</tr>
<tr>
<td>0.2</td>
<td>3404</td>
<td>3269</td>
<td>3240</td>
<td>3171</td>
<td>3138</td>
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<tr>
<td>0.4</td>
<td>10145</td>
<td>9639</td>
<td>9454</td>
<td>9147</td>
<td>9021</td>
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<tr>
<td>0.6</td>
<td>16691</td>
<td>15795</td>
<td>15310</td>
<td>14631</td>
<td>14359</td>
</tr>
<tr>
<td>0.8</td>
<td>23140</td>
<td>21582</td>
<td>20623</td>
<td>19687</td>
<td>19279</td>
</tr>
</tbody>
</table>

### Table 4.6: Mezzanine tranche spreads (bps per annum)

<table>
<thead>
<tr>
<th>RR</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>867.760</td>
<td>647.447</td>
<td>541.348</td>
<td>428.197</td>
<td>404.861</td>
</tr>
<tr>
<td>0.2</td>
<td>834.552</td>
<td>631.804</td>
<td>535.839</td>
<td>438.918</td>
<td>416.397</td>
</tr>
<tr>
<td>0.4</td>
<td>760.895</td>
<td>614.462</td>
<td>514.072</td>
<td>442.497</td>
<td>429.603</td>
</tr>
<tr>
<td>0.6</td>
<td>714.779</td>
<td>588.438</td>
<td>485.942</td>
<td>447.104</td>
<td>437.780</td>
</tr>
<tr>
<td>0.8</td>
<td>657.350</td>
<td>546.561</td>
<td>454.548</td>
<td>450.515</td>
<td>445.845</td>
</tr>
</tbody>
</table>
Chapter 4. Collateralized debt obligation pricing under copula framework

<table>
<thead>
<tr>
<th>RR</th>
<th>R</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.115</td>
<td>21.510</td>
<td>41.109</td>
<td>58.865</td>
<td>96.075</td>
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</tr>
<tr>
<td>0.2</td>
<td>0.093</td>
<td>17.632</td>
<td>34.180</td>
<td>48.996</td>
<td>80.141</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.071</td>
<td>13.449</td>
<td>26.384</td>
<td>38.291</td>
<td>62.934</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.047</td>
<td>9.123</td>
<td>18.082</td>
<td>26.547</td>
<td>43.789</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.024</td>
<td>4.661</td>
<td>9.302</td>
<td>13.775</td>
<td>22.811</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4.7: Senior tranche spreads (bps per annum)

There are different relationships between prices and correlations for different CDO tranches. Both the equity tranche and the mezzanine tranche show a negative relation between prices and correlations, while the opposite is true for the senior tranche. With regard to the recovery rates, both the senior tranche and the mezzanine tranche spreads decrease monotonically with the rise of the recovery rates, while the opposite is true for the equity tranche.

In other words, for the senior tranche, the spreads increase with an increase in correlation and a decrease in recovery rate. The converse is true for the equity tranche. As for the mezzanine tranche, the behavior is similar to the equity tranche in the relationship between the prices and correlations, while the behavior is similar to the senior tranche in the relationship between the prices and recovery rates. Although the results seem to be counterintuitive at first glance, they can be explained reasonably by analyzing the shape of loss distribution of the collateral pool under different assumptions of the correlation and the recovery rate referred to Boscher and Ward (2002).

The first two moments of the loss distribution can be given by

$$
\mathbb{E}[L(t)] = \sum_{i=1}^{n} (1 - RR_i)A_i1_{\{\tau_i \leq t\}};
\tag{4.42}
$$

$$
\mathbb{V}[L(t)] = \sum_{i=1}^{n} \mathbb{V}[L_i(t)] + \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \sqrt{\mathbb{V}[L_i(t)]\mathbb{V}[L_j(t)]},
\tag{4.43}
$$

where $i \neq j$ and $\mathbb{V}[L_i(t)] = (1 - RR_i)^2A_i^21_{\{\tau_i \leq t\}}(1 - 1_{\{\tau_i \leq t\}})$. For a homogeneous collateral pool, each reference entity has the same recovery rate $RR$ and notional amount $A$. The pairwise correlation is constant $r_{ij} = r$ and the credit default curve
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Figure 4.3: Equity tranche (0% - 3%)

Figure 4.4: Mezzanine tranche (3% - 14%)
is flat. Then (4.43) can be rewritten as

$$
\mathbb{V}[L(t)] = n\mathbb{V}[L_i(t)] + n(n - 1)r\mathbb{V}[L_i(t)].
$$

(4.44)

Therefore, the unexpected loss $UL$, which equals to the square root of the portfolio loss variance $\mathbb{V}[L(t)]$, is

$$
UL = \sqrt{\mathbb{V}[L(t)]} = \sqrt{n + n(n - 1)r(1 - RR)}\sqrt{1_{\{\tau_i \leq t\}}(1 - 1_{\{\tau_i \leq t\}})}.
$$

(4.45)

It is obvious that the loss distribution of the collateral pool is an increasing function of the correlation $r$ and a decreasing function of the recovery rate $RR$. More specifically, when the recovery rate decreases, the unexpected loss increases. A higher unexpected loss shifts the loss distribution from the centre to the tails, thus increasing the probability for losses in the senior tranche but decreasing the probability of losses in the equity tranche.
4.5.2 CDO pricing with different copulas

Dow Jones iTraxx Europe is based on a reference portfolio of 125 equally-weighted European investment grade companies. Figure 4.6 presents the CDS-bond basis and the trend of both iBoxx and iTraxx Europe from September 2015 to the end of 2018 provided by IHS Markit. It is clear that the spreads widened significantly in 2018, notably in the latter half after the European Central Bank (ECB) in June stated its intention to wind down the corporate sector purchase programme (CSPP) by the end of the year.

Consider the Dow Jones iTraxx Europe tranches with 5-year maturity from 5th of January 2015 to 10th of May 2016. The equity tranche is quoted as a market stand upfront premium plus 500 bps running. The recovery rate and risk-free interest rate are fixed to be 40% and 5% respectively. The hazard rate is therefore 0.025 according to table 4.3. LHP assumption is applied. Table 4.8 shows the attachment and detachment points of six tranches for iTraxx. By the simplex downhill solver and the gradient method, we can calibrate the parameters to fit the equity tranche and minimize the absolute error sum between the model implied spreads and the market spreads across all tranches. Six copulas described in subchapter 4.3 will be analyzed, namely Gaussian copula, double $t$ copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings, and Clayton copula.
Chapter 4. Collateralized debt obligation pricing under copula framework

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Attachment</th>
<th>Detachment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Mezzanine junior</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Mezzanine senior</td>
<td>60</td>
<td>90</td>
</tr>
<tr>
<td>Senior junior</td>
<td>90</td>
<td>120</td>
</tr>
<tr>
<td>Senior</td>
<td>120</td>
<td>220</td>
</tr>
<tr>
<td>Super senior</td>
<td>220</td>
<td>1 000</td>
</tr>
</tbody>
</table>

Table 4.8: Attachment and detachment points for iTraxx

Figure 4.7: Comparison of the compound correlation (left) and the base correlation (right)

Table 4.9 and figure 4.8 summarize the base correlation for the 5-year iTraxx Europe, which is derived from the implied compound correlation presented in table 4.9. The present value of each tranche can be valued as the difference between the present values of two base tranches. More specifically, for example, the present value of the 3%-6% tranche is equal to the difference between the present value of 0%-6% tranche and the present value of 0%-3% tranche. The base correlation and the compound correlation are equivalent for the 0%-3% tranche. The base correlation matches the price of the tranche and usually shows an upward sloping skew, while the compound correlation appears a correlation smile. The graphical comparison of the compound correlation and the base correlation is shown in figure 4.7. For the base correlation, the five lines from top to bottom represent the five tranches respectively. It is clear that the peak point occurs in May 2015 for all tranches. The average base correlation before May 2015 is higher than that after May 2015.
## Table 4.9: Compound correlation and base correlation for the 5-year iTraxx Europe

<table>
<thead>
<tr>
<th>Dates</th>
<th>Compound correlation</th>
<th>Base correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0%-3%</td>
<td>3%-6%</td>
</tr>
<tr>
<td>05/01/2015</td>
<td>20.5%</td>
<td>9.2%</td>
</tr>
<tr>
<td>01/02/2015</td>
<td>20.5%</td>
<td>8.3%</td>
</tr>
<tr>
<td>02/03/2015</td>
<td>20.9%</td>
<td>7.5%</td>
</tr>
<tr>
<td>06/04/2015</td>
<td>18.7%</td>
<td>7.9%</td>
</tr>
<tr>
<td>04/05/2015</td>
<td>21.7%</td>
<td>11.1%</td>
</tr>
<tr>
<td>01/06/2015</td>
<td>13.4%</td>
<td>13.5%</td>
</tr>
<tr>
<td>06/07/2015</td>
<td>14.5%</td>
<td>13.6%</td>
</tr>
<tr>
<td>03/08/2015</td>
<td>16.3%</td>
<td>12.8%</td>
</tr>
<tr>
<td>07/09/2015</td>
<td>11.8%</td>
<td>10.7%</td>
</tr>
<tr>
<td>05/10/2015</td>
<td>12.7%</td>
<td>12.2%</td>
</tr>
<tr>
<td>02/11/2015</td>
<td>12.4%</td>
<td>12.3%</td>
</tr>
<tr>
<td>07/12/2015</td>
<td>15.0%</td>
<td>13.9%</td>
</tr>
<tr>
<td>04/01/2016</td>
<td>13.1%</td>
<td>12.1%</td>
</tr>
<tr>
<td>01/02/2016</td>
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<td>10.6%</td>
</tr>
<tr>
<td>01/03/2016</td>
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<td>11.4%</td>
</tr>
<tr>
<td>05/04/2016</td>
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<td>14.5%</td>
</tr>
<tr>
<td>10/05/2016</td>
<td>14.2%</td>
<td>13.3%</td>
</tr>
</tbody>
</table>
Chapter 4. Collateralized debt obligation pricing under copula framework

The main functions in R are similar to what we used in the previous study, including the functions of cash flow and tranche spread, except the generation of the default time. Table 4.10 partially presents the ability of different copula models to fit the market quotes and correlation skew. Table 4.11 concludes the best fit parameters for each copula model. More results can be found in Appendix B.3.

Figure 4.9 shows the absolute error, which is computed by the difference between the model implied spreads and the market spreads, for each tranche. The equity tranche spreads are well fitted because the models are calibrated by the equity tranche. Although the Gaussian copula and the Clayton copula are quite different in their properties, their ability to fit the market quotes are very close for any tranche. Besides, the performances of both the Gaussian copula and the Clayton copula are obviously worse than the performances of other four copula models, especially for two mezzanine tranches. The peaks of the absolute errors for all tranches occur around May 2015.

The trend of the absolute error of the double $t$ copula is similar to that of the Gaussian copula and the Clayton copula for two mezzanine tranche, but slightly better than them. However, for two senior tranches, the movements of the absolute error of the double $t$ copula are in the approximately opposite direction against
that of the Gaussian copula and the Clayton copula. The average absolute error of the double $t$ copula decreases from the mezzanine tranches to the senior tranches, namely from 22.8 bps to 2.5 bps. The results illustrate that the double $t$ copula is good at fitting the senior tranches, especially the 12%-22% tranche, due to its characteristic of fat tails.

The NIG copula, Gaussian copula with stochastic correlation, and Gaussian copula with random factor loadings can fit the market quotes well in general. For the 3%-6% tranche, the absolute error of the RFL model ranges from 0 bps to 1.9 bps with an average of 0.2 bps, while the absolute error of the NIG model ranges from 0 bps to 6.5 bps with an average of 1 bps. The stochastic correlation model provides the absolute error ranges from 1.8 bps to 7 bps with an average of 4.6 bps, which is
also a good fit. The minimal average absolute error for the 6%-9% tranche is 5.5 bps by the NIG model, while the minimal average absolute error for the 9%-12% tranche is 3 bps by the RFL model. The stochastic correlation model usually provides the results between the results of the NIG model and the RFL model.

Figure 4.10 summarizes the absolute error sum of each copula model. It is obvious that the ability to fit the market quotes of the Gaussian copula and the Clayton copula are quite similar and both of them underperform the other four copula models. The highest absolute sum occurs in May 2015 with 224.6 bps by the Gaussian copula and the 225.5 bps by the Clayton copula. The trend of the absolute error sum of the double $t$ copula looks similar to that of the Gaussian copula and the Clayton copula. The absolute error sum of the double $t$ copula ranges from 9.5 bps to 76.1 bps with an average of 39.5 bps, which is very volatile. The rest three advanced copula models are all the extensions of the Gaussian copula, but they can provide relatively lower and more stable absolute error sum. The lowest absolute error sum is 4.1 bps.

<table>
<thead>
<tr>
<th>Models</th>
<th>Parameters</th>
<th>01/03/2016</th>
<th>05/04/2016</th>
<th>10/5/2016</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\rho = 10.9%$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Double $t$</td>
<td>$\rho = 13.2%, \nu = 3, \bar{\nu} = 7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NIG</td>
<td>$\alpha = 0.262, \beta = 0.220, \rho = 11.7%$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stoc.Corr.</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>RFL</td>
<td>$\rho_1 = 51.1%, \rho_2 = 4.6%, \theta = -2.56$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clayton</td>
<td>$\theta = 0.05$</td>
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<td></td>
<td></td>
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</table>

<table>
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<th>Models</th>
<th>Parameters</th>
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<th></th>
</tr>
</thead>
<tbody>
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<td></td>
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<tr>
<td>Double $t$</td>
<td>$\rho = 19.6%, \nu = 3, \bar{\nu} = 3$</td>
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<td></td>
</tr>
<tr>
<td>NIG</td>
<td>$\alpha = 0.255, \beta = 0.164, \rho = 16.6%$</td>
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<tr>
<td>Stoc.Corr.</td>
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<tr>
<td>RFL</td>
<td>$\rho_1 = 71.6%, \rho_2 = 5.1%, \theta = -2.50$</td>
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<td></td>
</tr>
<tr>
<td>Clayton</td>
<td>$\theta = 0.069$</td>
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<table>
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<tr>
<th>Models</th>
<th>Parameters</th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td></td>
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</tr>
<tr>
<td>Double $t$</td>
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<td></td>
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<tr>
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<td>$\rho = 9.0%, \eta = 22.5%, p = 0.746$</td>
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</tr>
<tr>
<td>RFL</td>
<td>$\rho_1 = 46.2%, \rho_2 = 4.8%, \theta = -2.38$</td>
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<tr>
<td>Clayton</td>
<td>$\theta = 0.0642$</td>
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</tbody>
</table>

Table 4.11: Best fit model parameters
Figure 4.9: Absolute error for each tranche
Figure 4.10: Absolute error sum

by the NIG model. The averages of the absolute error sum of the NIG model, the stochastic correlation model, and the RFL model are 16.2 bps, 27.8 bps, and 20.0 bps respectively.

4.6 Summary

This chapter starts from the basic structure of CDO and the pricing of CDO under copula framework. During the derivation of the CDO spread, it is not difficult to find that the core for pricing CDO is to determine the cumulative loss distribution function \( F_\infty(x) \), which leads to the development of the factor copula models. There are two main types of the factor copula models, namely the additive factor copulas and Archimedean copulas. Seven factor copula models are described in detail, including Gaussian copula, \( t \) copula, double \( t \) copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings, and Clayton copula. In order to derive an analytical situation for the approximating distribution of the portfolio loss, the large homogeneous portfolio approximation (LHP) is introduced as well.
We conduct two analyses, one is the analysis of how the tranche spreads of a CDO will change with different correlations and recovery rates, another is the ability of selected advanced factor copula models to fit the market quotes and correlation skew. In the first analysis, both the correlations and recovery rates are given from 0 to 0.8 with the increment of 0.2, and the default time is generated by the multinomial Gaussian copula with varying correlation matrices. There are only three tranches for simplification. The tranche spreads for each tranche are summarized in a matrix with 5 rows and 5 columns. The relationships between CDO spreads and correlations are different for different tranches. The CDO spreads increase with an increase in correlation and a decrease in recovery rate for the senior tranche, while the converse is true for the equity tranche. For the mezzanine tranche, the performance is similar to the equity tranche in the relationship between the spreads and correlations, while the performance is similar to the senior tranche in the relationship between the spreads and recovery rates. These results can be explained by the shape of the loss distribution, which is an increasing function of the correlation and a decreasing function of the recovery rate under certain assumptions.

The second analysis aims to compare the ability of different advanced factor copula models to fit the market quotes and correlation skew. The selected copula models include Gaussian copula, double $t$ copula, NIG copula, Gaussian copula with stochastic correlation, Gaussian copula with random factor loadings, and Clayton copula. The Dow Jones iTraxx Europe tranches with 5-year maturity from 5th of January 2015 to 10th of May 2016 is used as the input data. The recovery rate and the risk-free interest rate are uniformly set at 40% and 5% respectively. Six copula models, based on the LHP assumption, are analyzed and compared according to the absolute error obtained by the difference between the model implied spreads and the market spreads. The results of the Gaussian copula and the Clayton copula are quite similar for any tranche, and these two copula models underperform other four copula models. The trend of the absolute error of the double $t$ copula looks like that of both the Gaussian copula and the Clayton copula. The remaining three extensions of the Gaussian copula provide relatively better results. More specifically, the absolute error sum of NIG copula, Gaussian copula with stochastic correlation, and Gaussian copula with random factor loadings are lower and more stable. Besides, the lowest absolute error sum is 4.1 bps given by the NIG model.
Chapter 5

Dynamic copula model

Since the dependence structure in financial markets is usually nonlinear, asymmetric, and dynamic, it is necessary to estimate a dynamic model to describe this kind of dependence structure. Erb et al. (1994), Longin and Solnik (2001), and Ang and Chen (2002) report that two asset returns exhibit greater correlation during market downturns than market upturns. One of the outstanding features of the financial time series is the existence of the conditional heteroscedasticity. Engle (1982) proposes the autoregressive conditional heteroscedasticity (ARCH) to describe the conditional second moment of the financial time series, and use the change of the ARCH to reflect the time-varying and clustering fluctuation. Most popular ARCH processes include ARCH by Engle (1982), GARCH by Bollerslev (1986), IGARCH by Engle and Bollerslev (1986), EGARCH by Nelson (1990), and so on. Overview of GARCH models can be found in Bollerslev et al. (1992 and 1994), Shephard (1996), Gouriéroux (1997), and Francq and Zakoïan (2010).

Patton (2001, 2004 and 2006), Fortin and Kuzmics (2002), Chen and Fan (2006), Jondeau and Rockinger (2006), Ausin and Lopes (2010), and Creal et al. (2013) develop the dynamic time-series models based on copulas where the copula functional form is fixed while its parameters are allowed to vary through time as a function of the lagged information, which is similar to the ARCH model for volatility. Hamilton (1989) introduces the Markov regime switching model (MRS) and Hamilton and Susmel (1994) combines MRS and ARCH models. For the conditional copula, the functional form of the copula is allowed to vary through time considered by Rodriguez (2007), Okimoto (2008), Chollette et al. (2008), Markwat et al. (2008), and Garcia and Tsafack (2009).

In this chapter, we firstly introduce the copula-ARCH model, including copula-GARCH model, copula-IGARCH model, and copula-EGARCH model. Then we
turn to the time-varying copula model to form a dynamic conditional joint distribution. Two typical time-varying copula models include the time-varying Gaussian copula model and the time-varying symmetrized Joe-Clayton (SJC) copula model. Besides, an algorithm named Markov regime switching model (MRS) is proposed to model the changes in regime. The combination of MRS and ARCH process is the MRS-ARCH(\(q\)) model, which is extended to the MRS-copula-GARCH(\(k,p,q\)) later. In the empirical study, we conduct three studies to verify and find the best copula model for the asset portfolio distribution. The first two focus on the comparison of the constant copula, the time-varying copula, and the time-varying copula with Markov regime switching in terms of their negative log-likelihood. The lower the negative log-likelihood, the better performance of the copula model. The third study is about the dependence of constituents of the FTSE 100 Index by the time-varying copula-GARCH model. What’s more, in order to test the goodness-of-fit of the copula models, a powerful test named the Hit test is applied.

### 5.1 Multivariate copula-ARCH model

The existence of the conditional variance is an important feature of financial time series. Engle (1982) introduces a new class of stochastic processes called ARCH (autoregressive conditional heteroscedasticity) processes, which are mean zero, serially uncorrelated with nonconstant variances conditional on the past but constant unconditional variances. After that, numerous extensions have been proposed to forecast risk measures.

We will start from the multivariate copula-ARCH model. Assume \(N\) random sequences \(\{y_{1t}\}_{t=1}^{T}, \{y_{2t}\}_{t=1}^{T}, \ldots, \{y_{Nt}\}_{t=1}^{T}\), the multivariate copula-ARCH(\(q\)) is

\[
y_{nt} = \mu_{nt} + \varepsilon_{nt}, n = 1, 2, \ldots, N, t = 1, 2, \ldots, T, \tag{5.1a}
\]
\[
\varepsilon_{nt} = h_{nt}^{1/2} \xi_{nt}, \tag{5.1b}
\]
\[
h_{nt} = \omega_n + \sum_{i=1}^{q_n} \alpha_{ni} \varepsilon_{n,t-i}^2, \tag{5.1c}
\]
\[
(\xi_{1t}, \xi_{2t}, \ldots, \xi_{Nt}) \sim C_t(\Phi(\xi_{1t}), \Phi(\xi_{2t}), \ldots, \Phi(\xi_{Nt})), \tag{5.1d}
\]

where \(\xi_{nt} \sim i.i.N(0, 1)\). Moreover, (5.1d) can be rewritten as

\[
(\xi_{1t}, \xi_{2t}, \ldots, \xi_{Nt}) \sim C_t(F_1(\xi_{1t}), F_2(\xi_{2t}), \ldots, F_N(\xi_{Nt})) \tag{5.2}
\]
where $\xi_{nt} \sim i.i.d.(0, 1)$ and $F_n(\cdot)$ is the normal distribution with a mean of 0 and a standard deviation of 1.

### 5.1.1 Copula-GARCH model

Based on the theories of copula and GARCH (generalized ARCH), copula-ARCH model can be extended to copula-GARCH model. Nelson (1990) derives the condition for strict stationarity of GARCH models in the case of the GARCH(1,1) model and Bougerol and Picard (1992) generalizes this to GARCH($p,q$). Assume $N$ random sequences $\{y_{1t}\}_{t=1}^{T}, \{y_{2t}\}_{t=1}^{T}, \cdots, \{y_{Nt}\}_{t=1}^{T}$, the multivariate copula-GARCH($p,q$) is

$$y_{nt} = \mu_{nt} + \varepsilon_{nt}, \quad n = 1, 2, \cdots, N, \ t = 1, 2, \cdots, T, \quad \text{(5.3a)}$$

$$\varepsilon_{nt} = h_{nt}^{1/2} \xi_{nt}, \quad \text{(5.3b)}$$

$$h_{nt} = \omega_n + \sum_{i=1}^{q_n} \alpha_{ni} \varepsilon_{n,t-i}^2 + \sum_{i=1}^{p_n} \beta_{ni} h_{n,t-i}, \quad \text{(5.3c)}$$

$$(\xi_{1t}, \xi_{2t}, \cdots, \xi_{Nt}) \sim C_t(F_1(\xi_{1t}), F_2(\xi_{2t}), \cdots, F_N(\xi_{Nt})), \quad \text{(5.3d)}$$

where $\xi_{nt} \sim N(0, 1)$.

### 5.1.2 Copula-IGARCH model

The GARCH process is weakly stationary because the mean, variance, and autocovariance are finite and constant over time. Nelson (1990) proves that the IGARCH (integrated GARCH) process with a positive drift is strictly stationary and ergodic. The multivariate copula-IGARCH is

$$y_{nt} = \mu_{nt} + \varepsilon_{nt}, \quad n = 1, 2, \cdots, N, \ t = 1, 2, \cdots, T, \quad \text{(5.4a)}$$

$$\varepsilon_{nt} = h_{nt}^{1/2} \xi_{nt}, \quad \text{(5.4b)}$$

$$\phi_n(L)(1 - L)\varepsilon_{nt}^2 = \alpha_{n0} + (1 - \beta_n(L))\nu_{nt}, \quad \text{(5.4c)}$$

$$(\xi_{1t}, \xi_{2t}, \cdots, \xi_{Nt}) \sim C_t(F_1(\xi_{1t}), F_2(\xi_{2t}), \cdots, F_N(\xi_{Nt})), \quad \text{(5.4d)}$$

where $\phi_n(L)(1 - L) = 1 - \alpha_n(L) - \beta_n(L)$ and $\nu_{nt} = \varepsilon_{nt}^2 - h_{nt}$. 

5.1.3 Copula-EGARCH model

Nelson and Cao (1992) argue that the nonnegativity constraints in the linear GARCH model are too restrictive, while there are no restrictions in the EGARCH (exponential GARCH) model. In the EGARCH model, the conditional variance $h_t$ is an asymmetric function of lagged disturbances $\varepsilon_{t-1}$:

\begin{align*}
  y_{nt} &= \mu_{nt} + \varepsilon_{nt}, n = 1, 2, \cdots, N, t = 1, 2, \cdots, T, \\
  \varepsilon_{nt} &= h_{nt}^{1/2} \xi_{nt}, \\
  \ln h_{nt} &= \alpha_n + \sum_{k=1}^{\infty} \beta_{nk} g_n(\xi_{nt-k}) + \sum_{j=1}^{q_n} \alpha_{nj} \ln h_{n,t-j}, \\
  (\xi_{1t}, \xi_{2t}, \cdots, \xi_{Nt}) &\sim C_t(F_1(\xi_{1t}), F_2(\xi_{2t}), \cdots, F_N(\xi_{Nt})),
\end{align*}

and

\begin{equation}
  g_n(\xi_{nt}) = \theta_{n} \xi_{nt} + \gamma_n(|\xi_{nt}| - E(|\xi_{nt}|)),
\end{equation}

where $g_n(\cdot)$ satisfies $E_{t-1}(g_n(\xi_{nt})) = 0$.

5.2 Time-varying copula model

Time-varying copula models are introduced by Patton (2001) for the first time to form a dynamic conditional joint distribution. There are two typical time-varying copula models, including the bivariate time-varying Gaussian copula model and the bivariate time-varying symmetrized Joe-Clayton (SJC) copula model. The normal distribution is symmetric and tail asymptotically independent; while the SJC copula can capture the asymmetry and tail dependence.

5.2.1 Bivariate time-varying Gaussian copula model

Recall that bivariate Gaussian copula is

\begin{equation}
  C(u, v; \rho) = \int_{-\infty}^{\varphi^{-1}(u)} \int_{-\infty}^{\varphi^{-1}(v)} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp[-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}]dsdt,
\end{equation}

where $\varphi^{-1}$ is the inverse of the standard normal cumulative density function and $\rho \in (-1, 1)$ is the correlation coefficient. $\rho$ can be either the constant or time-varying.
Patton (2006) proposes that the upper and lower tail dependence parameters each follow something akin to a restricted ARMA(1,10) process:

$$\rho_t = \tilde{\Lambda}(\omega + \beta \cdot \rho_{t-1} + \alpha \cdot \frac{1}{10} \sum_{i=1}^{10} \Phi^{-1}(u_{t-i}) \cdot \Phi^{-1}(v_{t-i})),$$

(5.8)

where \(\tilde{\Lambda} = (1 - e^{-x})(1 + e^{-x})^{-1} = \tanh(x/2)\) is the modified logistic transformation designed to maintain \(\rho_t \in (-1, 1)\) and \(\{u_t\}_{t=1}^T, \{v_t\}_{t=1}^T\) are the sequences based on probability integral transformation of the observed sequences. \(\rho_{t-1}\) is the regressor to capture any persistence in the dependence parameter, and the mean of the product of the last 10 observations of the transformed variables \(\Phi^{-1}(u_{t-i})\) and \(\Phi^{-1}(v_{t-i})\) to capture the variation in dependence. Specifically speaking, if the product of \(\Phi^{-1}(u_{t-i})\) and \(\Phi^{-1}(v_{t-i})\) is positive, the point \((\Phi^{-1}(u_{t-i}), \Phi^{-1}(v_{t-i}))\) lies in the first or third quadrant; if the product of \(\Phi^{-1}(u_{t-i})\) and \(\Phi^{-1}(v_{t-i})\) is negative, the point \((\Phi^{-1}(u_{t-i}), \Phi^{-1}(v_{t-i}))\) lies in the second or fourth quadrant.

### 5.2.2 Bivariate time-varying SJC copula model

The Joe-Clayton copula, referred to the BB7 copula, is constructed by taking a particular Laplace transformation of Clayton copula. More details can be found in Joe (1997). The distribution function of the bivariate Joe-Clayton copula is

$$C_{JC}(u, v; \tau_U, \tau_L) = 1 - \left(\left[1 - (1 - u)\kappa\right]^{-\gamma} + \left[1 - (1 - v)\kappa\right]^{-\gamma} - 1\right)^{-\frac{1}{\gamma}},$$

(5.9)

where \(\kappa = 1/\log_2(2 - \tau_U)\) and \(\gamma = -1 \log_2(\tau_L).\) Two parameters, \(\tau_U \in (0, 1)\) and \(\tau_L \in (0, 1),\) are dependence measures known as tail dependence. The Joe-Clayton copula allows upper and lower tail dependence to range anywhere from zero to one. Figure 5.1 plots an example of the density plot and contour plot of distributions defined with Joe-Clayton copula with standard normal margins and \(\kappa = 1.42, \gamma = 0.47.\)

Although the two tail dependence measures are equal, there is still some slight asymmetry in the Joe-Clayton copula because of its functional form. Therefore, Patton (2001) proposes the symmetrized Joe-Clayton (SJC) copula, which is symmetric when \(\tau_U = \tau_L:\)

$$C_{SJC}(u, v; \tau_U, \tau_L) = 0.5 \cdot C_{JC}(u, v; \tau_U, \tau_L) + C_{JC}(1-u, 1-v; \tau_U, \tau_L) + u + v - 1.$$  

(5.10)
The evolution equations for the symmetrized Joe-Clayton copula are:

\[
\tau_t^U = \Lambda(\omega_U + \beta_U \tau_{t-1}^U) + \alpha_U \cdot \frac{1}{10} \sum_{i=1}^{10} |u_{t-i} - v_{t-i}|, \\
\tau_t^L = \Lambda(\omega_L + \beta_L \tau_{t-1}^U) + \alpha_L \cdot \frac{1}{10} \sum_{i=1}^{10} |u_{t-i} - v_{t-i}|, 
\]

(5.11) (5.12)

where \(\Lambda(x) = (1 + e^{-x})^{-1}\) is the logistic transformation to maintain \(\tau_U, \tau_L \in (0, 1)\).

The right hand side of the model for the tail dependence evolution equation contains an autoregressive term, \(\beta_U \tau_{t-1}^U\) and \(\beta_L \tau_{t-1}^L\), and a forcing variable.

It is somewhat difficult to identify a forcing variable, so we can use the mean absolute difference between \(u_t\) and \(v_t\) over the previous 10 observations as a forcing variable, which can be explained in figure 5.2. If \(X\) and \(Y\) are perfectly positively dependent, the transformed variables \(U\) and \(V\) will lie on the main diagonal of the unit square. Because \(|u_t - v_t|\) is proportional to the minimum distance from the point \((u_t, v_t)\) to the main diagonal, it is reasonable to use the mean absolute difference between \(u_t\) and \(v_t\) over the previous 10 observations as an indication of the distance between the data and the perfectly positive dependence.

Since the upper and lower tail dependence measures are one-to-one functions of the two parameters in SJC copula, \(\gamma_t\) and \(\kappa_t\) can be computed by

\[
\gamma_t = \gamma(\tau_L^t) = -[\log_2(\tau_L^t)]^{-1}, \\
\kappa_t = \kappa(\tau_U^t) = -[\log_2(\tau_U^t)]^{-1}.
\]

(5.13)
\[ \kappa_t = \kappa(\tau^U_t) = \left[ \log_2(2 - \tau^U_t) \right]^{-1}. \]  

\section*{5.3 Markov regime switching-copula-GARCH model}

Hamilton (1989) points out that the business cycle has a recurrent feature of periodic shifts between a recessionary state and a growth state. In order to model the changes in regime, an algorithm named Markov regime switching model (MRS) is proposed for drawing such probabilistic inference in the form of a nonlinear iterative filter. The main idea of MRS is to use unobserved state variables and the transition between states is governed by a Markov process. Later, Hamilton and Susmel (1994) combines MRS and ARCH models by allowing the parameters of an ARCH process to come from one of several different regimes, with transitions between regimes governed by an unobserved Markov chain.

Suppose a random process \( \{y_t\} \), then MRS-ARCH(\(q\)) is

\[ y_t = \mu + \phi y_{t-1} + \varepsilon_t, \]  
\[ \varepsilon_t = \sqrt{h_t(s_t)} \xi_t, \xi_t \sim i.i.N(0, 1), \]  
\[ h_t(s_t) = \omega_1 s_t + \omega_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \]
where \( s_t \) is unobserved state, which can be either 0 or 1, and the transition between states is assumed to be governed by a first-order Markov process:

\[
\begin{align*}
P(s_t = 1 | s_{t-1} = 1) &= p, & (5.16a) \\
P(s_t = 0 | s_{t-1} = 1) &= 1 - p, & (5.16b) \\
P(s_t = 0 | s_{t-1} = 0) &= q, & (5.16c) \\
P(s_t = 1 | s_{t-1} = 0) &= 1 - q. & (5.16d)
\end{align*}
\]

Due the fact that MRS-ARCH(\( q \)) is associated with a numerous parameters and the reality usually requires a higher-order process, which will lead to the problem of multicollinearity. Therefore, MRS-GARCH(\( k, p, q \)) comes into being. MRS-GARCH(\( k, p, q \)) is

\[
\begin{align*}
y_t &= a + y_{t-1} + u_{s_t}, & (5.17a) \\
u_{s_t} &= \sqrt{g_{s_t}} \xi_t, & (5.17b) \\
\xi_t &= \sqrt{h_t(s_t)} \xi_t, \xi_t \sim i.i.N(0, 1), & (5.17c) \\
h_t &= \omega + \sum_{i=1}^{q} \alpha_i \xi_{t-i}^2 + \sum_{j=1}^{p} \beta_i h_{t-j}, & (5.17d)
\end{align*}
\]

where \( s_t = 0, 1, \ldots, k \) is unobserved state and \( g_{s_t} \) is the degree of influence on the volatility caused by the state \( s_t \). Moreover, the transition between states obeys a discrete Markov process:

\[
p_{ij} = P(s_t = j | s_{t-1} = i, s_{t-2} = k, \ldots) & (5.18)
\]

So the matrix of transition probability is

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1k} \\
p_{21} & p_{22} & \cdots & p_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k1} & p_{k2} & \cdots & p_{kk}
\end{bmatrix}, \tag{5.19}
\]

where \( \sum_{j=1}^{k} p_{ij} = 1, 0 \leq p_{ij} \leq 1, \) and \( i, j = 1, 2, \ldots, k. \)

Rodriguez (2007) models nonlinearity and asymptotic dependence by use of copulas combined with Markov switching parameters to study financial contagion. The
model can be denoted by MRS-copula-GARCH($k, p, q$) for simplification. Generally, MRS-copula-GARCH($2, 1, 1$) is effective enough to describe the marginal distribution of the financial sequences. Consider two financial sequences $\{y_{nt}\}_{t=1}^{T}, n = 1, 2$, the bivariate MRS-copula-GARCH($2, 1, 1$) is

\[ y_{nt} = a_n + y_{n,t-1} + u_{nt}, t = 1, \cdots, T, \]
\[ u_{nt} = \sqrt{g_{nst}} \varepsilon_{nt}, \]
\[ \varepsilon_{nt} = \sqrt{h_{nt}} \xi_{nt}, \xi_{nt} \sim i.i.N(0, 1), \]
\[ h_{nt} = \omega_n + \alpha_n \varepsilon_{n,t-1}^2 + \beta_n h_{n,t-1}. \]

Assume that the dependence between the financial sequence $\{y_{1t}\}_{t=1}^{T}$ and $\{y_{2t}\}_{t=1}^{T}$ is different under different switching regimes, then the copula functions in the bivariate MRS-copula-GARCH($2, 1, 1$) is

\[ (\xi_{1t}, \xi_{2t}) \sim C_t(\Phi(\xi_{1t}), \Phi(\xi_{2t}); \kappa(s_t)), \]

where $C_t(\cdot, \cdot)$ is a bivariate copula function and $\kappa(s_t)$ is the parameter vector at the unobserved state.

It is convenient to estimate the parameters of the density by maximum likelihood estimation, because a bivariate density can be decomposed into the product of the copula and the univariate marginals. Assuming a two-state Markov chain for each sequence, the likelihood of each observation is

\[ q_t(x_t, y_t|I_{t-1}; \theta) = \sum_{s_t} \sum_{s_{t-1}} f_t(x_t|s_t, s_{t-1}, I_{t-1}; \theta) \times g_t(y_t|s_t, s_{t-1}, I_{t-1}; \theta) \]
\[ \times C_t(u_t, v_t|s_t, s_{t-1}, I_{t-1}; \theta) \times P(s_t, s_{t-1}|I_{t-1}; \theta), \]

where $\theta$ is a vector of parameters, $f, g,$ and $C$ are the marginals and copula densities respectively, and $P(s_t, s_{t-1}|I_{t-1}; \theta)$ is the probability of the state. Besides, $u_t = F_t(x_t|s_t, s_{t-1}, I_{t-1}; \theta)$ and $v_t = G_t(x_t|s_t, s_{t-1}, I_{t-1}; \theta)$, where $F$ and $G$ are the distribution functions of $x_t$ and $y_t$ respectively. Therefore, the likelihood function to be maximized is

\[ L(\theta) = \sum_{t=1}^{T} \ln [q_t(x_t, y_t|I_{t-1}; \theta)] \]

Although it is straightforward to fit the marginal distributions, the dependence
parameter estimation through copulas requires additional process because $\theta$ depends on a unobserved discrete state $s_t$ following a Markov chain. Kim and Nelson (1999) describes a so-called Kim’s filter to proceed with the estimation process. The prediction of $s_t$ is

$$P(s_t = l|I_{t-1}) = \sum_{k=0}^{1} p_{kl}^{t-1} P(s_{t-1} = k|I_{t-1}),$$

(5.24)

where $l = 0, 1$ and $p_{kl}^{t-1} = P(s_t = l|s_{t-1} = k, I_{t-1})$ is the transition probabilities between the states $k$ and $l$. Then the filtering of $s_t$ is

$$P(s_t = l|I_t) = \frac{c_t(u, v|s_t = l, I_{t-1})P(s_t = l|I_{t-1})}{\sum_{k=0}^{1} c_t(u, v|s_t = k, I_{t-1})P(s_t = k|I_{t-1})},$$

(5.25)

where $I_t = [I_{t-1}, u_t, v_t]$. When $t = 1$, the filter is initialized using the stationary probability of $s_t$ for $P(s_0 = k|I_0)$.

### 5.4 Empirical study

There are three analyses in the empirical study. The first study is the comparison of the constant and the time-varying copula. We will consider two stocks issued by ExxonMobil and IBM in Frankfurt Stock Exchange (FSE), and the time horizon is from 11\textsuperscript{th} of May 2009 to 15\textsuperscript{th} of March 2019 to have 2,500 observations for each stock. In the comparison, eight constant copulas and three time-varying copulas are selected. The eight constant copulas include Gaussian copula, $t$ copula, Gumbel copula, rotated Gumbel copula, Clayton copula, rotated Clayton copula, Frank copula, and symmetrized Joe-Clayton (SJC) copula; while the three time-varying copulas include time-varying Gaussian copula, time-varying rotated Gumbel copula, and time-varying SJC copula. The eleven copulas are ranked in terms of the negative log-likelihood. The one with the lowest log-likelihood is the optimal copula. Besides, the eleven copulas can also be ranked by AIC and BIC to make sure that the number of parameters is considered according to (2.20) and (2.21).

The second study aims to apply the time-varying copula with Markov regime switching to the same two stocks in order to compare with the previous eleven copulas. The three time-varying copulas are extended to the Markov regime switching
version and are assumed to be two regimes. Then the fourteen copulas in total can be ranked according to their negative log-likelihood.

The third study focuses on the dependence of constituents of the FTSE 100 Index by the time-varying copula-GARCH model. Four stocks in the FTSE 100 Index are analyzed, including EXPN.L (Experian plc), RMV.L (Rightmove plc), CNA.L (Centrica plc), and CCH.L (Coca-Cola HBC AG), from 6th of June 2013 to 13th of May 2019 to have 1 500 observations for each stock. The time-varying copula-GARCH(1,1)-t model is selected to fit the sample data. Then we can estimate the marginal parameters of the four stocks and the parameters of the desired copula models, including the time-varying $t$ copula, the static SJC copula, and the time-varying SJC copula. Lastly, a powerful test, which is so-called the Hit test, will be conducted to test the goodness-of-fit of the copula models.

### 5.4.1 Comparison of the constant and the time-varying copula for a two-asset portfolio

Consider two stocks issued by ExxonMobil and IBM in Frankfurt Stock Exchange (FSE) from 11th of May 2009 to 15th of March 2019. There are 2 500 observations for each stock. We firstly use the empirical cumulative distribution function to transform the data in MATLAB.

```matlab
uiopen('xon_ibm.txt',1)
[xon,ibm] = textread('xon_ibm.txt');
u = empiricalcdf(xon);
v = empiricalcdf(ibm);
T = length(u)
T = 2500

function out1 = empiricalcdf(data,xx);
% Computes empirical probability integral transform
[T,k] = size(data);
if nargin<2
    out1 = -999.99*ones(T,k);
for jj = 1:k
    temp = [data(:,jj),(1:1:T)’];
    temp2 = sortrows(temp,1);
    temp3 = [temp2,(1:1:T)/(T+1)]; % dividing by T+1 rather than T so that none equals 1
    out1(:,jj) = temp3(:,3);
end
else
    [q,q2] = size(xx);
    if q2=1
        xx = xx*ones(1,k);
    end
    out1 = -999.99*ones(q,k);
for jj = 1:k
    for ii = 1:q
```

```matlab
```
```
Chapter 5. Dynamic copula model

We create the optimization options structure by the function \textit{optimset}. Eight constant copulas are chosen to be compared, including Gaussian copula, \textit{t} copula, Gumbel copula, rotated Gumbel copula, Clayton copula, rotated Clayton copula, Frank copula, and symmetrized Joe-Clayton (SJC) copula. The MATLAB code for the negative copula log-likelihood and cumulative distribution function of these eight copulas can be found in Appendix C.1. We refer to \textit{Patton Copula Toolbox} for more details.

```matlab
% estimating some copula models
options = optimset('Display','iter','TolCon',10^-12,'TolFun',10^-4,'TolX',10^-6);

% 1. Gaussian copula
kappa1 = corrcoef(norminv(u),norminv(v));
LL1 = NormalCopula_CL(kappa1,[u,v]);

% 2. Student's t copula
lower = [-0.9 , 2.1 ];
upper = [ 0.9 , 100 ];
theta0 = [kappa1;10];
[kappa2 LL2] = fmincon('tcopulaCL',theta0,[],[],[],[],lower,[],[],options,[u,v]);

% 3. Gumbel copula
lower = 1.1;
theta0 = 2;
[kappa3 LL3] = fmincon('GumbelCL',theta0,[],[],[],[],lower,[],[],options,[u,v]);

% 4. Rotated Gumbel copula
lower = 1.1;
theta0 = 2;
[kappa4 LL4] = fmincon('GumbelCL',theta0,[],[],[],[],lower,[],[],options,1-[u,v]);

% 5. Clayton's copula
lower = 0.0001;
theta0 = 1;
[kappa5 LL5] = fmincon('ClaytonCL',theta0,[],[],[],[],lower,[],[],options,[u,v]);

% 6. Rotated Clayton copula (with tail dependence in upper tail instead of lower)
lower = 0.0001;
theta0 = 1;
[kappa6 LL6] = fmincon('ClaytonCL',theta0,[],[],[],[],lower,[],[],options,1-[u,v]);

% 7. Frank copula
theta0 = 1;
[kappa7 LL7] = fmincon('FrankCL',theta0,[],[],[],[],lower,[],[],options,[u,v]);

% 8. Symmetrized Joe-Clayton copula
lower = [0 , 0 ];
upper = [ 1 , 1 ];
theta0 = [0.250.25];
[kappa8 LL8] = fmincon('sym_jc_CL',theta0,[],[],[],[],lower,upper,[],options,[u,v]);

LL = [LL1;LL2;LL3;LL4;LL5;LL6;LL7;LL8];
[1:length(LL)'
LL] sortrows([1:length(LL)'
LL],2)
ans =

1.0000 -43.8661
2.0000 -63.9544
3.0000 -44.5847
```
Chapter 5. Dynamic copula model

The optimal copula in terms of log-likelihood is the one with the lowest likelihood, which means the minimal negative log-likelihood. It is obvious that the best copula is $t$-copula, followed by SJC copula; while the worst and the second-worst are rotated Clayton copula and Clayton copula respectively.

We can also have a look at the tail dependence implied by each of these eight copulas. Table 5.1 summarizes the results. Gaussian copula has zero tail dependence. $t$ copula has symmetric tail dependence. Gumbel copula has zero lower tail dependence and rotated Gumbel copula has zero upper tail dependence. Clayton copula has zero upper tail dependence and rotated Clayton copula has zero lower tail dependence. Frank copula has zero tail dependence. SJC copula parameters are the tail dependence coefficients, but in reverse order.

tauLU = nines(8,2);
tauLU(1,:) = [0,0]; % Gaussian copula has zero tail dependence
tauLU(2,:) = ones(1,2)*2*tdis_cdf(-sqrt((kappa2(2)+1)*(1-kappa2(1))/(1+kappa2(1))),kappa2(2)+1); % $t$ copula has symmetric tail dependence
tauLU(3,:) = [0,2-2^(1/kappa3)]; % Gumbel copula has zero lower tail dependence
tauLU(4,:) = [2-2^(1/kappa4),0]; % Rotated Gumbel copula has zero upper tail dependence
tauLU(5,:) = [2^(-1/kappa5),0]; % Clayton copula has zero upper tail dependence
tauLU(6,:) = [0,2^(-1/kappa6)]; % Rotated Clayton copula has zero lower tail dependence
tauLU(7,:) = [0,0]; % Frank copula has zero tail dependence
tauLU(8,:) = kappa8([2,1])'; % SJC copula parameters are the tail dependence coefficients, but in reverse order.
tauLU

sortrows([1:8]',LL,tauLU,2)
tauLU =

ans =

2.0000 -63.9544
8.0000 -56.5927
4.0000 -55.6489
3.0000 -44.5847
1.0000 -43.8661
7.0000 -42.8903
5.0000 -42.3799
6.0000 -35.1564
Table 5.1: Constant copulas ranked by the log-likelihood

<table>
<thead>
<tr>
<th>Copula</th>
<th>Log-likelihood</th>
<th>Upper tail dependence</th>
<th>Lower tail dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>-63.9544</td>
<td>0.0447</td>
<td>0.0447</td>
</tr>
<tr>
<td>SJC</td>
<td>-56.5927</td>
<td>0.0799</td>
<td>0.0347</td>
</tr>
<tr>
<td>Rotated Gumbel</td>
<td>-55.6489</td>
<td>0.1591</td>
<td>0</td>
</tr>
<tr>
<td>Gumbel</td>
<td>-44.5847</td>
<td>0</td>
<td>0.1462</td>
</tr>
<tr>
<td>Gaussian</td>
<td>-43.8661</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Frank</td>
<td>-42.8903</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Clayton</td>
<td>-42.3799</td>
<td>0.0457</td>
<td>0</td>
</tr>
<tr>
<td>Rotated Clayton</td>
<td>-35.1564</td>
<td>0</td>
<td>0.0317</td>
</tr>
</tbody>
</table>

Now we can extend constant copulas to time-varying copulas, including time-varying Gaussian copula, time-varying rotated Gumbel copula, and time-varying SJC copula. Then it is possible to rank the eleven copulas in total. The MATLAB code for the negative copula log-likelihood of these three copulas can be found in Appendix C.1. In the new ranking, the 1st is time-varying SJC copula, the 2nd is time-varying rotated Gumbel copula, the 3rd is constant t copula, and the 4th is time-varying Gaussian copula. The results illustrate that the time-varying copulas outperform the constant copulas in most cases.
Chapter 5. Dynamic copula model

Figure 5.3, figure 5.4, and figure 5.5 continuously plot the conditional tail dependence estimates from Gaussian copula, rotated Gumbel copula, and SJC copula respectively. Figure 5.3 looks nice, while figure 5.4 does not because the variation fluctuates like noise. In figure 5.5, the movement in upper tail dependence looks like noise as well, whereas the movement in lower tail dependence is not so dramatically fluctuant.

Moreover, the eleven copulas can be ranked by AIC and BIC. The rankings by AIC and BIC are the same as by log-likelihood as the results show. Table 5.2 summarizes the results we have obtained.
Chapter 5. Dynamic copula model

**Figure 5.3:** Conditional tail dependence in Gaussian copula

**Figure 5.4:** Conditional tail dependence in rotated Gumbel copula
Chapter 5. Dynamic copula model

**Figure 5.5:** Conditional tail dependence in SJC copula

<table>
<thead>
<tr>
<th>Value</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
<th>Parameter 3</th>
<th>Parameter 4</th>
</tr>
</thead>
<tbody>
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<td>-70.9928</td>
<td>-141.9808</td>
<td>-141.9669</td>
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<tr>
<td>10.0000</td>
<td>-66.5178</td>
<td>-133.0263</td>
<td>-133.0263</td>
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<td>-127.9056</td>
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<td>-111.2946</td>
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<tr>
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**Figure 5.5:** Conditional tail dependence in SJC copula

<table>
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<th>Parameter 3</th>
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<td>-133.0263</td>
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<td>-127.9056</td>
<td>-127.9056</td>
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<tr>
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<td>-58.6740</td>
<td>-117.3386</td>
<td>-117.3386</td>
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<tr>
<td>6.0000</td>
<td>-35.1564</td>
<td>-70.3097</td>
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</tbody>
</table>

**Figure 5.5:** Conditional tail dependence in SJC copula

<table>
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<th>Value</th>
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<th>Parameter 2</th>
<th>Parameter 3</th>
<th>Parameter 4</th>
</tr>
</thead>
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<td>-141.9669</td>
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</tr>
<tr>
<td>10.0000</td>
<td>-66.5178</td>
<td>-133.0263</td>
<td>-133.0263</td>
<td></td>
</tr>
<tr>
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<td>-63.9544</td>
<td>-127.9056</td>
<td>-127.9056</td>
<td></td>
</tr>
<tr>
<td>9.0000</td>
<td>-58.6740</td>
<td>-117.3386</td>
<td>-117.3386</td>
<td></td>
</tr>
<tr>
<td>8.0000</td>
<td>-56.5927</td>
<td>-113.1792</td>
<td>-113.1792</td>
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</tr>
<tr>
<td>4.0000</td>
<td>-55.6489</td>
<td>-111.2946</td>
<td>-111.2946</td>
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</tr>
<tr>
<td>3.0000</td>
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<td>-89.1687</td>
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<tr>
<td>1.0000</td>
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<td>-87.7314</td>
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<td>-85.7743</td>
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<tr>
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<td>6.0000</td>
<td>-35.1564</td>
<td>-70.3097</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter 5. Dynamic copula model

5.4.2 Application of the time-varying copula with Markov regime switching for a two-asset portfolio

Furthermore, we can generate the negative log-likelihood of the time-varying copula with Markov regime switching, which is assumed to be two regimes in our empirical study. The input data is the same as what we used in the last subchapter for convenience for the comparison. We continuously extend time-varying Gaussian, rotated Gumbel, and SJC copula to time-varying Markov copulas, and then compare these new-added results with previous results. The MATLAB code for the negative copula log-likelihood of these three time-varying copulas with Markov regime switching, presented in Appendix C.1, can be referred to Markov Copula Toolbox.

% 12. Time-varying Gaussian Markov copula
theta0=[1 5 0 0 4 5];
[param12 LL12] = fmincon('markovnormal_LLF',theta0,[],[],[],[],lower,upper,[],options,[],[u,v],kappa1);
[LL12, CL12, theta_s0N, theta_s1N, pf1N, pf2N, pplus1N, pplus2N] = markovnormal_LLF(param12,[u,v],kappa1);

% 13. Time-varying rotated Gumbel Markov copula
theta0=[1 5 0 0 4 5];
[param13 LL13] = fmincon('markovgumbel_LLF',theta0,[],[],[],[],lower,upper,[],options,[1-u,1-v],kappa3);
[LL13, CL13, theta_s0GL, theta_s1GL, pf1GL, pf2GL, pplus1GL, pplus2GL] = markovgumbel_LLF(param13,[1-u,1-v],kappa4);

% 14. Time-varying SJC Markov copula
lower = -30*ones(10,1);
upper = 30*ones(10,1);
theta0=[1 1 -1 -1 0 0 -1 -1 4 5];
[param14 LL14] = fmincon('markovsjc_LLF',theta0,[],[],[],[],lower,upper,[],options,[u,v],kappa8);

<table>
<thead>
<tr>
<th>Copula</th>
<th>Log-likelihood</th>
<th>AIC</th>
<th>BIC</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric copulas</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>-43.8661</td>
<td>-87.7314</td>
<td>-87.7290</td>
<td>8</td>
</tr>
<tr>
<td>Time-varying Gaussian</td>
<td>-58.6740</td>
<td>-117.3456</td>
<td>-117.3386</td>
<td>4</td>
</tr>
<tr>
<td>t</td>
<td>-63.9544</td>
<td>-127.9079</td>
<td>-127.9056</td>
<td>3</td>
</tr>
<tr>
<td>Frank</td>
<td>-42.8903</td>
<td>-85.7789</td>
<td>-85.7743</td>
<td>9</td>
</tr>
<tr>
<td>SJC</td>
<td>-56.5927</td>
<td>-113.1839</td>
<td>-113.1792</td>
<td>5</td>
</tr>
<tr>
<td>Time-varying SJC</td>
<td>-70.9928</td>
<td>-141.9808</td>
<td>-141.9669</td>
<td>1</td>
</tr>
<tr>
<td>Asymmetric copulas</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gumbel</td>
<td>-44.5847</td>
<td>-89.1687</td>
<td>-89.1664</td>
<td>7</td>
</tr>
<tr>
<td>Rotated Gumbel</td>
<td>-55.6489</td>
<td>-111.2969</td>
<td>-111.2946</td>
<td>6</td>
</tr>
<tr>
<td>Time-varying rotated Gumbel</td>
<td>-66.5178</td>
<td>-133.0333</td>
<td>-133.0263</td>
<td>2</td>
</tr>
<tr>
<td>Clayton</td>
<td>-42.3799</td>
<td>-84.7591</td>
<td>-84.7568</td>
<td>10</td>
</tr>
<tr>
<td>Rotated Clayton</td>
<td>-35.1564</td>
<td>-70.3120</td>
<td>-70.3097</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 5.2: Results from constant and time-varying copulas
Up to now, the 1st is time-varying rotated Gumbel Markov copula, the 2nd is time-varying Gaussian Markov copula, the 3rd is time-varying SJC copula, then followed by time-varying SJC Markov copula, time-varying rotated Gumbel copula, \( \tau \) copula, and time-varying Gaussian copula. Table 5.3 summarizes the results of 14 copulas in total. The results effectively clarify that the time-varying copulas with Markov regime switching can perform even better on the basis of the time-varying copulas. However, consider the fact that it is more complicated to model a time-varying copula with Markov regime switching than a time-varying copula or a static copula, we can choose the appropriate copula model based on the specific input data and objectives in reality.
Table 5.3: Ranking of the copulas

<table>
<thead>
<tr>
<th>Ranking</th>
<th>Copula</th>
<th>Log-likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Time-varying rotated Gumbel Markov copula</td>
<td>-81.0947</td>
</tr>
<tr>
<td>2</td>
<td>Time-varying Gaussian Markov copula</td>
<td>-77.2252</td>
</tr>
<tr>
<td>3</td>
<td>Time-varying SJC copula</td>
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</tr>
<tr>
<td>4</td>
<td>Time-varying SJC Markov copula</td>
<td>-70.9863</td>
</tr>
<tr>
<td>5</td>
<td>Time-varying rotated Gumbel copula</td>
<td>-66.5178</td>
</tr>
<tr>
<td>6</td>
<td>$t$ copula</td>
<td>-63.9544</td>
</tr>
<tr>
<td>7</td>
<td>Time-varying Gaussian copula</td>
<td>-58.6740</td>
</tr>
<tr>
<td>8</td>
<td>SJC copula</td>
<td>-56.5927</td>
</tr>
<tr>
<td>9</td>
<td>Rotated Gumbel copula</td>
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<tr>
<td>10</td>
<td>Gumbel copula</td>
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</tr>
<tr>
<td>11</td>
<td>Gaussian copula</td>
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</tr>
<tr>
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<td>Clayton copula</td>
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</tr>
<tr>
<td>14</td>
<td>Rotated Clayton copula</td>
<td>-35.1564</td>
</tr>
</tbody>
</table>

5.4.3 Analysis of the FTSE 100 index by the time-varying copula-GARCH model

In order to study the dependence of constituents of the FTSE 100 Index, 4 stocks listed in the FTSE 100 Index are selected, including EXPN.L (Experian plc), RMV.L (Rightmove plc), CNA.L (Centrica plc), and CCH.L (Coca-Cola HBC AG). The data is collected from Yahoo Finance during the period from June 6th, 2013 to May 13th, 2019 to have 1,500 observations. Table 5.4 summarizes the descriptive statistics of the returns. Returns of EXPN.L and CCH.L are right-tailed, while returns of RMV.L and CCH.L are left-tailed. All four returns are excess kurtosis. According to the Jarque-Bera test, the null hypothesis of normal distribution is rejected for all returns at the 5% significance level.

The conditional distribution of financial time series is time-varying, volatility clustering, leptokurtosis, and fat-tailed. The $t$ distribution is usually good at describing the distribution of financial time series. In this case, we choose the copula-GARCH(1,1)-$t$ model to fit the sample data. Specifically, a bivariate time-varying
### Table 5.4: Descriptive statistics of the returns of four stocks

<table>
<thead>
<tr>
<th></th>
<th>EXPN.L</th>
<th>RMV.L</th>
<th>CNA.L</th>
<th>CCH.L</th>
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<tbody>
<tr>
<td>Mean</td>
<td>0.0004848</td>
<td>0.0007949</td>
<td>-0.0007650</td>
<td>0.0004148</td>
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<tr>
<td>Median</td>
<td>0.0005735</td>
<td>0.0007765</td>
<td>0</td>
<td>0.0004199</td>
</tr>
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<td>St. deviation</td>
<td>0.0125</td>
<td>0.0159</td>
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</tr>
<tr>
<td>IQR</td>
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<td>0.0149</td>
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<td>Skewness</td>
<td>0.0191</td>
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<td>Kurtosis</td>
<td>6.2535</td>
<td>12.8509</td>
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<td>Jarque-Bera test</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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</table>

The copula-GARCH(1,1)-t model integrating (5.3) and (5.8) is

\[
R_{nt} = c_0 + c_1 R_{n,t-1} + \varepsilon_{nt}, \quad n = 1, 2, t = 1, 2, \ldots, T, \tag{5.26a}
\]

\[
\varepsilon_{nt} = h_{nt}^{1/2} \xi_{nt}, \tag{5.26b}
\]

\[
h_{nt} = \omega_n + \alpha_n \varepsilon_{n,t-1}^2 + \beta_n h_{n,t-1}, \tag{5.26c}
\]

\[
(\xi_{1t}, \xi_{2t}) \sim C_{Ga}(t_{\nu_1}(\xi_{1t}), t_{\nu_2}(\xi_{2t}); \rho_t), \tag{5.26d}
\]

\[
\rho_t = \tilde{\Lambda}(\omega_{\rho} + \beta_{\rho} \cdot \rho_{t-1} + \alpha_{\rho} \cdot \frac{1}{10} \sum_{i=1}^{10} \Phi^{-1}(u_{t-i}) \cdot \Phi^{-1}(v_{t-i})), \tag{5.26e}
\]

where \( C_{Ga} \) is the bivariate Gaussian copula, \( t_{\nu_1} \) and \( t_{\nu_2} \) are \( t \) distribution with the mean of 0, the standard deviation of 1, the degree of freedom of \( \nu_1 \) and \( \nu_2 \) respectively, and \( u_t = t_{\nu_1}(\xi_{1t}), v_t = t_{\nu_2}(\xi_{2t}) \).

We can use Dynamic Copula Toolbox to estimate the copula-GARCH(1,1)-t model in MATLAB. Firstly, the GARCH model for each series can be estimated by the function `modelspec` to make the desired choices for the margins. There will be four pop-up windows as presented in figure 5.6. Just clicking GARCH model for each series, GARCH(1,1), \( T \), and IFM continuously.

```matlab
```

Then the estimated parameters can be obtained by the function `fitModel`. Besides, LogLM is the vector of the marginal log likelihoods at the optimum, evalM is a cell that contains structures, GradHessM is a cell that contains structures, and udata is the matrix of sample GARCH residuals turned to uniform. Figure 5.7 presents two pop-up windows in this function. We can use input defaults to define the starting values and choose yes to calculate the asymptotic standard error.
**Chapter 5. Dynamic copula model**

![Pop-up windows for specifying the margins](image)

**Figure 5.6: Pop-up windows for specifying the margins**

---

<table>
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<th>parameter</th>
<th>St. Error</th>
<th>t-stats</th>
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<td>1.4199</td>
</tr>
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<td>0.000</td>
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</tr>
<tr>
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<td>0.021</td>
<td>1.7221</td>
</tr>
<tr>
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</tr>
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</table>

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<td>0.000</td>
<td>1.9066</td>
</tr>
<tr>
<td>0.1304</td>
<td>0.042</td>
<td>3.0595</td>
</tr>
<tr>
<td>0.8038</td>
<td>0.072</td>
<td>11.1844</td>
</tr>
<tr>
<td>3.7776</td>
<td>0.416</td>
<td>9.0710</td>
</tr>
</tbody>
</table>

Akaike: -9050.4003  
BIC: -9018.5210  
Log Likelihood: 4531.200

Estimation time is 27.90 seconds
Press any key to continue

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<td>0.000</td>
<td>1.0354</td>
</tr>
<tr>
<td>0.0357</td>
<td>0.021</td>
<td>1.7221</td>
</tr>
<tr>
<td>0.9368</td>
<td>0.045</td>
<td>20.8528</td>
</tr>
<tr>
<td>5.4986</td>
<td>0.761</td>
<td>7.2247</td>
</tr>
</tbody>
</table>

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<th>t-stats</th>
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<td>0.061</td>
<td>-0.2092</td>
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<td>1.9066</td>
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<td>0.1304</td>
<td>0.042</td>
<td>3.0595</td>
</tr>
<tr>
<td>0.8038</td>
<td>0.072</td>
<td>11.1844</td>
</tr>
<tr>
<td>3.7776</td>
<td>0.416</td>
<td>9.0710</td>
</tr>
</tbody>
</table>

Akaike: -8594.6173  
BIC: -8562.7380  
Log Likelihood: 4303.309

---

Computing finite-difference Hessian using objective function.

Estimation output

Parameter | St. Error | t-stats |
----------|-----------|---------|
0.0025    | 0.000     | 1.8164  |
0.0390    | 0.027     | 1.4199  |
0.0000    | 0.000     | 1.0354  |
0.0357    | 0.021     | 1.7221  |
0.9368    | 0.045     | 20.8528 |
5.4986    | 0.761     | 7.2247  |

Akaike: -9050.4003  
BIC: -9018.5210  
Log Likelihood: 4531.200

---

Computing finite-difference Hessian using objective function.

Parameter | St. Error | t-stats |
----------|-----------|---------|
0.0011    | 0.000     | 3.5224  |
-0.0127   | 0.061     | -0.2092 |
0.0000    | 0.000     | 1.9066  |
0.1304    | 0.042     | 3.0595  |
0.8038    | 0.072     | 11.1844 |
3.7776    | 0.416     | 9.0710  |

Akaike: -8594.6173  
BIC: -8562.7380  
Log Likelihood: 4303.309

---

Press any key to continue.
Chapter 5. Dynamic copula model

The results are summarized in Table 5.5. All the parameters are corresponded to (5.26). The Kolmogorov-Smirnov test illustrates that it fails to reject the null hypothesis that the data comes from a standard normal distribution at the 5% significance level for all margins, which means the sequence obtained by the probability integral transformation based on the conditional marginal distribution estimated by the GARCH(1,1)-t model obeys i.i.d. (0,1) distribution. Thus, the GARCH(1,1)-t model
can well fit the conditional marginal distribution.

For the \( t \) distribution, the shape of the distribution plot is highly correlated with the degree of freedom \( \nu \). It has fatter tails if the degree of freedom is smaller. From the results in table 5.5, the degrees of freedom are different for the four observations and the smallest is provided by CCH.L, which means the return of CCH.L is more fat-tailed. Typically, a bivariate distribution requires that the two variables have the identical marginal distribution. In the reality, however, it is difficult to fit a certain bivariate distribution when the marginal distributions are different. In our cases, the degrees of freedom are different, although they are all \( t \) distributed. Thus, it is not reasonable to simply assumed that their joint distribution is a bivariate \( t \) distribution. Since the copula models have no restricts of the choices of the marginal distribution, they can be flexibly applied into the analysis of financial time series.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>EXPN.L</th>
<th>RMVL</th>
<th>CNA.L</th>
<th>CCH.L</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 )</td>
<td>0.0005 ( (0.000) )</td>
<td>0.0011 ( (0.000) )</td>
<td>-0.0002 ( (0.000) )</td>
<td>0.0005 ( (0.000) )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>0.0390 ( (0.027) )</td>
<td>-0.0127 ( (0.061) )</td>
<td>0.0053 ( (0.010) )</td>
<td>-0.0061 ( (0.023) )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>4.2030 ( \cdot 10^{-6} ) ( (0.000) )</td>
<td>2.0147 ( \cdot 10^{-6} ) ( (0.000) )</td>
<td>2.4628 ( \cdot 10^{-6} ) ( (0.000) )</td>
<td>9.4927 ( \cdot 10^{-6} ) ( (0.000) )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.0357 ( (0.021) )</td>
<td>0.1304 ( (0.042) )</td>
<td>0.0237 ( (0.010) )</td>
<td>0.0681 ( (0.032) )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.9368 ( (0.045) )</td>
<td>0.8038 ( (0.072) )</td>
<td>0.9685 ( (0.015) )</td>
<td>0.9081 ( (0.048) )</td>
</tr>
<tr>
<td>( \nu )</td>
<td>5.4986 ( (0.761) )</td>
<td>3.7776 ( (0.416) )</td>
<td>3.4056 ( (0.352) )</td>
<td>3.2424 ( (0.273) )</td>
</tr>
<tr>
<td>AIC</td>
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<td>-8594.6173</td>
<td>-8641.2283</td>
<td>-8553.9443</td>
</tr>
<tr>
<td>BIC</td>
<td>-9018.5210</td>
<td>-8562.7380</td>
<td>-8609.3490</td>
<td>-8522.0650</td>
</tr>
<tr>
<td>LL</td>
<td>-4531.200</td>
<td>-4303.309</td>
<td>-4326.614</td>
<td>-4282.972</td>
</tr>
<tr>
<td>K-S test</td>
<td>0.4797</td>
<td>0.4747</td>
<td>0.4772</td>
<td>0.4746</td>
</tr>
</tbody>
</table>

Table 5.5: Estimated marginal parameters of time-varying copula-GARCH(1,1)-\( t \) model for 4 stocks

Once the marginal parameters are known, the function \texttt{modelspec} is used again to specify the desired copula from three pop-up windows as shown in figure 5.8. Clicking \texttt{Copula} first and then defining the time-varying Clayton copula, the static SJC copula, and the time-varying SJC copula respectively. Note that \texttt{udata} in \texttt{modelspec} here is the matrix of sample GARCH residuals turned to uniform, which we have obtained from the last step. Moreover, the estimated parameters for the three
desired copula models can be obtained by the function `fitModel`. The two pop-up windows are the same as those in figure 5.7.

```matlab
specC = modelSpec(udata);
[parsC, LogLC, evalC, GradHessC] = fitModel(specC, udata, 'fminunc');

Computing finite-difference Hessian using objective function.

<table>
<thead>
<tr>
<th>parameter</th>
<th>St. Error</th>
<th>t-stats</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0690</td>
<td>0.164</td>
<td>-0.4212</td>
</tr>
<tr>
<td>-1.1521</td>
<td>0.633</td>
<td>-1.8209</td>
</tr>
<tr>
<td>0.4257</td>
<td>0.190</td>
<td>2.2418</td>
</tr>
</tbody>
</table>

Akaike: -197.9087
BIC: -181.9690
Log Likelihood: 101.954

Estimation time is 21.37 seconds

Computation finite-difference Hessian using objective function.

<table>
<thead>
<tr>
<th>parameter</th>
<th>St. Error</th>
<th>t-stats</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3495</td>
<td>0.048</td>
<td>7.3504</td>
</tr>
<tr>
<td>0.3352</td>
<td>0.045</td>
<td>7.4893</td>
</tr>
</tbody>
</table>

Akaike: -253.8404
BIC: -243.2139
Log Likelihood: 128.920

Estimation time is 3.79 seconds

Computation finite-difference Hessian using objective function.

<table>
<thead>
<tr>
<th>parameter</th>
<th>St. Error</th>
<th>t-stats</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8985</td>
<td>0.631</td>
<td>1.4244</td>
</tr>
<tr>
<td>-4.6891</td>
<td>3.273</td>
<td>-1.4328</td>
</tr>
<tr>
<td>0.4381</td>
<td>0.287</td>
<td>1.5252</td>
</tr>
<tr>
<td>-0.2807</td>
<td>0.237</td>
<td>-1.1870</td>
</tr>
<tr>
<td>1.1582</td>
<td>0.961</td>
<td>1.2058</td>
</tr>
<tr>
<td>0.8802</td>
<td>0.101</td>
<td>8.7357</td>
</tr>
</tbody>
</table>
```

**Figure 5.8:** Pop-up windows for the estimated parameters
Table 5.6 summarizes the estimated parameters in (5.8) for the three desired copula models. For the time-varying SJC copula, the left and right column corresponds to the coefficients of the upper and lower tail dependence respectively. The parameter $\beta_\rho$ is high, thus the correlation coefficient $\rho_t$ at time $t$ is closely correlated with the correlation coefficient $\rho_{t-1}$ at time $t - 1$. According to the value of the negative log-likelihood, the time-varying SJC copula model performs best because it provides the minimal negative log-likelihood. Besides, AIC and BIC suggest the same result and these two criteria have the advantage that they consider the number of parameters when choosing the best model.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Time-varying Clayton</th>
<th>SJC</th>
<th>Time-varying SJC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau^U$</td>
<td>0.3495 (0.048)</td>
<td>0.3352 (0.045)</td>
<td></td>
</tr>
<tr>
<td>$\tau^L$</td>
<td>0.8985 (0.631)</td>
<td>-0.2807 (0.237)</td>
<td></td>
</tr>
<tr>
<td>$\omega_\rho$</td>
<td>-0.0690 (0.164)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_\rho$</td>
<td>-1.1521 (0.633)</td>
<td>-4.6891 (3.273)</td>
<td>1.1582 (0.961)</td>
</tr>
<tr>
<td>$\beta_\rho$</td>
<td>0.4257 (0.190)</td>
<td>0.4381 (0.287)</td>
<td>0.8802 (0.101)</td>
</tr>
<tr>
<td>AIC</td>
<td>-197.9087</td>
<td>-253.8404</td>
<td>-251.0105</td>
</tr>
<tr>
<td>BIC</td>
<td>-181.9690</td>
<td>-243.2139</td>
<td>-219.1312</td>
</tr>
<tr>
<td>LL</td>
<td>-101.954</td>
<td>-128.920</td>
<td>-131.505</td>
</tr>
</tbody>
</table>

| TABLE 5.6: Estimated parameters of the copula models |

To compare the goodness-of-fit of the copula models, we can employ the Hit test proposed by Patton (2006), which extends the original Hit test presented in the Christoffersen (1998) and Engle and Manganelli (1999) to nonlinear density models by decomposing a density model into a set of region models. Figure 5.9 presents the regions used in the Hit test. Regions 1 and 2 represent the lower and upper 10% of the distribution, while Regions 3 and 4 represent observations are somewhere between the 10th and 25th quantiles or between 75th and 90th quantiles. Region 5 in the middle represent 50% of the distribution. Regions 6 and 7 are extremely asymmetric regions with the upper and lower 25% of the distribution. Each region should
be correctly specified under the null hypothesis. If every region model pass the test, then the entire density model is considered correctly specified.

Specify a simple linear $\lambda_j$ function $\lambda_j(Z_{jt}, \beta_j) = Z_{jt} \cdot \beta_j$, where the regressor $Z_{jt}$ is a constant. When $\beta_j = 0$, then $\pi_{jt} = \pi_j(Z_{jt}, \beta_j, p_{jt}) = \pi_j(Z_{jt}, 0, p_{jt}) = p_{jt}$, namely the probability estimated by the model is equal to the real probability. Therefore, the Hit test aims to test the hypothesis $H_0 : \beta_j = 0$ versus $H_1 : \beta_j \neq 0$. The null hypothesis indicates that the model is correctly specified in the region $R_j$.

Let’s define the hit variables to be analyzed in the test first. Note that violations is a series of violations (0,1), which is a $(T + \text{lagorder}, 1)$ series; VaRbdd is a series of VaR, which is also a $(T + \text{lagorder}, 1)$ series; and lagorder is the number of lags.

The function of the dynamic logistic model is defined to have a likelihood ratio test under the null hypothesis that the model is correctly specified in each region. In the function, $X$ is the explanatory variables, $T$ is the sample size, and $\text{risk}$ is the
alpha coverage rate. The estimation of the dynamic logistic model can be found in Appendix C.2, which is referred to Dumitrescu et al. (2012).

\[
\text{function } \left[ \text{res} \right] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{X}, \text{T}, \alpha) \\
K = \text{size} (\text{X}, 2) + 1; \quad \% \text{ Number of explicite variables + lagged index} \\
\text{\% Estimation under H0} \\
\text{}\text{const} = \log (\text{risk}/(1 - \text{risk})); \quad \% \text{ Fractile of the logistic distribution} \\
\text{LRH0} = \sum (\text{hit} .* \log (\text{risk}) + (1 - \text{hit}) .* \log (1 - \text{risk})); \quad \% \text{ Log-likelihood under H0} \\
\text{\% Estimation under H1} \\
[\text{resH1}] = \text{DL} (\text{hit}, \text{X}, \text{T}); \quad \% \text{ Estimation of dynamic logistic under H1} \\
\text{LRH1} = \text{resH1}\_\text{loglikelihood}; \quad \% \text{ Log-likelihood under H1} \\
\text{hitproba} = \text{resH1}\_\text{proba}; \quad \% \text{ hit probability - logistic model} \\
\text{LR} = -2\times(\text{LRH0} - \text{LRH1}); \quad \% \text{ LRT statistic} \\
\text{LR\_pvalue} = 1 - \text{chi2cdf} (\text{LR}, K); \quad \% \text{ p-value} \\
\]

Then we can define the function of backtesting for each region to obtain the test statistic and \(p\)-value. Any \(p\)-value less than alpha suggests a rejection of the null hypothesis that the model is well-specified. Alpha can be both 0.01 and 0.05 in the test. Table 5.7 summarizes the results for 7 regions.

\[
\text{function } \left[ \text{statfin}, \text{pvaluefin}, \text{proba}\right] = \text{Backtest} (\text{violations}, \text{VaRbdd}, \text{lagorder}, \alpha) \\
\%	ext{ Dynamic logistic specification 1} \\
\text{X} = \text{ones} (\text{T}, 1); \\
[\text{res}] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{X}, \text{T}, \alpha); \\
\text{res1} = \text{res}\_\text{LR\_pvalue}; \\
\text{exf1} = \text{res}\_\text{exitflag}; \\
\text{lr1} = \text{res}\_\text{LR}; \\
\text{proba1} = \text{res}\_\text{hitproba}; \\
\%	ext{ Dynamic logistic specification 2} \\
\text{XX} = [\text{ones} (\text{T}, 1) \text{HIT}(:, 2)]; \\
[\text{res}] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{XX}, \text{T}, \alpha); \\
\text{res2} = \text{res}\_\text{LR\_pvalue}; \\
\text{exf2} = \text{res}\_\text{exitflag}; \\
\text{lr2} = \text{res}\_\text{LR}; \\
\text{proba2} = \text{res}\_\text{hitproba}; \\
\%	ext{ Dynamic logistic specification 3} \\
\text{XX} = [\text{ones} (\text{T}, 1) \text{HIT}(:, 2) \text{HIT}(:, 3)]; \\
[\text{res}] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{XX}, \text{T}, \alpha); \\
\text{res3} = \text{res}\_\text{LR\_pvalue}; \\
\text{exf3} = \text{res}\_\text{exitflag}; \\
\text{lr3} = \text{res}\_\text{LR}; \\
\text{proba3} = \text{res}\_\text{hitproba}; \\
\%	ext{ Dynamic logistic specification 4} \\
\text{XX} = [\text{ones} (\text{T}, 1) \text{HIT}(:, 2) \text{HIT}(:, 3) \text{HIT}(:, 4)]; \\
[\text{res}] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{XX}, \alpha); \\
\text{res4} = \text{res}\_\text{LR\_pvalue}; \\
\text{exf4} = \text{res}\_\text{exitflag}; \\
\text{lr4} = \text{res}\_\text{LR}; \\
\text{proba4} = \text{res}\_\text{hitproba}; \\
\%	ext{ Dynamic logistic specification 5} \\
\text{XX} = [\text{ones} (\text{T}, 1) \text{HIT}(:, 2) \text{VaR}(:, 2)]; \\
[\text{res}] = \text{Logistic}\_\text{VaR}\_\text{DL} (\text{hit}, \text{XX}, \alpha); \\
\text{res5} = \text{res}\_\text{LR\_pvalue}; \\
\text{exf5} = \text{res}\_\text{exitflag}; \\
\text{lr5} = \text{res}\_\text{LR}; \\
\text{proba5} = \text{res}\_\text{hitproba}; \\
\]
% Dynamic logistic specification 6
XX=[ ones(T,1) HIT(:,2) VaR(:,2) VaR(:,2) ];
[res]=Logistic_VaR_DL(hit,XX,T,alpha);
res6=res.LR_pvalue;
exf6=res.exitflag;
lr6=res.LR;
proba6=res.hitproba;

% Dynamic logistic specification 7
XX=[ ones(T,1) VaR(:,2) ];
[res]=Logistic_VaR_DL(hit,XX,T,alpha);
res7=res.LR_pvalue;
exf7=res.exitflag;
lr7=res.LR;
proba7=res.hitproba;
pvaluefin=[res1 res2 res3 res4 res5 res6 res7];
statfin=[lrcc1 lrcc2 lrcc3 lrcc4 lrcc5 lrcc6 lrcc7];
proba=[proba1 proba2 proba3 proba4 proba5 proba6 proba7];
table=[statfin' pvaluefin'];

<table>
<thead>
<tr>
<th>Region</th>
<th>Time-varying Clayton</th>
<th>SJC</th>
<th>Time-varying SJC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.6888 (0.2156)**</td>
<td>7.8452 (0.0745)**</td>
<td>6.4568 (0.2978)**</td>
</tr>
<tr>
<td>2</td>
<td>0.2681 (0.1124)**</td>
<td>1.0510 (0.1469)**</td>
<td>1.6247 (0.1740)**</td>
</tr>
<tr>
<td>3</td>
<td>0.7556 (0.5805)**</td>
<td>5.0041 (0.5229)**</td>
<td>0.1650 (0.6204)**</td>
</tr>
<tr>
<td>4</td>
<td>1.7599 (0.7617)**</td>
<td>0.2069 (0.7872)**</td>
<td>0.4041 (0.8304)**</td>
</tr>
<tr>
<td>5</td>
<td>3.7786 (0.6422)**</td>
<td>0.7430 (0.5390)**</td>
<td>3.4782 (0.6897)**</td>
</tr>
<tr>
<td>6</td>
<td>0.5752 (0.7067)**</td>
<td>0.0890 (0.5297)**</td>
<td>0.6928 (0.5709)**</td>
</tr>
<tr>
<td>7</td>
<td>0.1194 (0.8152)**</td>
<td>2.3946 (0.3171)**</td>
<td>0.0831 (0.3636)**</td>
</tr>
</tbody>
</table>

Table 5.7: Hit test results for the copula models

Table 5.7 presents the LR test statistics and the p-value of three copula models for 7 regions. * means it is significant when alpha is 0.01, while ** means it is significant when alpha is 0.05. The results show that the three copula models easily pass all individual regions, except the p-value of the SJC copula model for the 1st region is close to 0.05. Therefore, these three copula models are well-specified, and two time-varying copula models slightly outperform the static copula model.

5.5 Summary

This chapter focuses on the dynamic copula models to describe the nonlinear, asymmetric, and dynamic dependence structure in financial markets. The first is the multivariate copula-ARCH model to describe the conditional variance of financial time series. The extensions include copula-GARCH model, copula-IGARCH model, and copula-EGARCH model. Time-varying copula models are usually used to form a dynamic conditional joint distribution, such as the time-varying Gaussian copula model and time-varying SJC copula model. The last is the introduction of the Markov regime switching model (MRS) to draw the probabilistic inference by a nonlinear iterative filter. The combination of MRS, ARCH, and copulas is so-called the MRS-copula-GARCH\((k,p,q)\) model.

The empirical study consists of three parts. The first study is the comparison of the constant or static copula models and the time-varying copula models. Two stocks issued by ExxonMobil and IBM in FSE are considered from 11\(^{th}\) of May 2009 to 15\(^{th}\) of March 2019. There are eight constant copulas and three time-varying copulas in the comparison, including Gaussian copula, \(t\) copula, Gumbel copula, rotated Gumbel copula, Clayton copula, rotated Clayton copula, Frank copula, SJC copula, time-varying Gaussian copula, time-varying rotated Gumbel copula, and time-varying SJC copula. These eleven copulas are ranked according to their negative log-likelihood and the optimal copula is the one with the lowest negative log-likelihood. The best one is the time-varying SJC copula. Besides, the result of ranking is identical when these copulas are ranked by AIC and BIC. In most cases, the time-varying copulas outperform the constant copulas.

In the second study, the time-varying copulas with Markov regime switching are applied to the same input data as in the first study for the convenient for the comparison. The MRS is assumed to be two regimes and three new added copulas include time-varying Markov Gaussian, rotated Gumbel, and SJC copula. The fourteen copulas in total can be ranked together based on the negative log-likelihood and the best one is the time-varying rotated Gumbel Markov copula with the lowest negative log-likelihood of -81.0947. The results suggest that the time-varying copulas with Markov regime switching can generally perform even better than the time-varying copulas. In reality, due to the fact that it is more complicated to model a time-varying copula with MRS than a time-varying copula or a constant copula, it is necessary to choose the appropriate copula model considering the specific input.
As for the third study, we focus on the dependence structure of the FTSE 100 Index by the time-varying copula-GARCH model. Four components of the FTSE 100 Index are chosen from 15\textsuperscript{th} of June 2012 to 26\textsuperscript{th} of July 2019, including EXPN.L (Experian plc), RMV.L (Rightmove plc), CNA.L (Centrica plc), and CCH.L (Coca-Cola HBC AG). A bivariate time-varying copula-GARCH(1,1)-t model is applied to fit the data since the conditional distribution of financial time series is usually time-varying, volatility clustering, leptokurtosis, and fat-tailed. The Kolmogorov-Smirnov test shows that the GARCH(1,1)-t model is able to well fit the conditional marginal distribution. Then we select the time-varying Clayton copula, the static SJC copula, and the time-varying SJC copula as the desired copula models. The results states that the correlation coefficient $\rho_t$ at time $t$ is closely correlated with the correlation coefficient $\rho_{t-1}$ at time $t-1$ because the parameter $\beta_\rho$ is high. Besides, the time-varying SJC copula model outperforms other two copula models because of its minimal negative log-likelihood. Furthermore, a powerful test named the Hit test is conducted to test the goodness-of-fit of the copula models. All three copula models pass the test easily, which means they are well-specified, and the time-varying Clayton copula and the time-varying SJC copula perform slightly better than the static SJC copula.
Chapter 6

Conclusion

Credit risk is omnipresent in financial markets, so it is important to accurately measure and manage credit risk. Since the dependence structure of financial time series is usually nonlinear, asymmetric, and time-varying, copula functions can well describe the dependence structure and tail dependence. The good properties of copula functions make them widely used in the dependence analysis and multivariate distribution of portfolio credit risk.

The thesis focuses on the application of copula functions in credit risk measurement and management. The main research works in the thesis can be summarized as follows:

1. The research status of the application of copula functions in Finance were systematically reviewed and the basic properties and theories of copula functions were comprehensively introduced both mathematically and graphically.

2. We described credit risk models briefly, including the structural and reduced-form models, and the CreditMetrics™ model in detail. Two different portfolios consisting of ten bonds traded on FSE were created, one was a good-quality portfolio, and another was a credit-risky portfolio. As summarized in subchapter 3.4, we calibrated the parameters of copula functions to fit the sample data and obtained the correlation matrix based on copulas. Besides, the implicit assumption of normal distribution was removed and Monte Carlo simulation was conducted based on the real distribution of the portfolio value. According to the comparison of the VaR by the original CreditMetrics™ model and the CreditMetrics™ model based on copulas of two portfolios, we found that copula functions helped improve the accuracy of measuring credit risk for the credit-risky portfolio at any confidence levels.
3. We introduced the structure and properties of a CDO and derived the CDO
pricing formula in the semi-analytic approach under copula framework. The
conditional probabilities of default and the approximating distributions of the
portfolio loss associated with selected factor copula models were summarized.
Furthermore, we figured out that the CDO spreads increased with an increase
in correlation and with a decrease in recovery rate for the senior tranche, while
the converse was true for the equity tranche. As for the mezzanine tranche, it
was similar to the equity tranche in the relationship between the spreads and
correlations, while it was similar to the senior tranche in the relationship be-
tween the spreads and recovery rates. After comparing the ability of selected
factor copula models to fit the market quotes and correlation skew, we found
that three extensions of the Gaussian copula model outperformed others and
the lowest absolute error sum was provided by the NIG model.

4. Dynamic copula models were presented, including multivariate copula-ARCH
model, time-varying copula model, and MRS-copula-GARCH model. Firstly,
we compared the negative log-likelihood of static copulas, time-varying cop-
ulas, and MRS-copula-GARCH copulas, and found that MRS-copula-GARCH
copulas usually provided the lower negative log-likelihood. Moreover, we
used the time-varying copula-GARCH models to study the dependence of
constituents of the FTSE 100 Index. We found that the time-varying SJC copula
was the best, which provided the lowest negative log-likelihood.

Apart from the achievements of the thesis mentioned above, some questions for
further research are put forward as follows:

1. The thesis only applies bivariate copula functions to the measurement of credit
risk and fails to study the multivariate copula functions in the reality. It is the
research emphasis in the future.

2. In the application of the CreditMetrics™ model, the measurement of credit
risk is easily affected by the rating system of the rating agency although the
introduction of copula functions is effective. Therefore, it is necessary to study
how to measure portfolio credit risk more accurately considering more infor-
mation about the specific financial situations of bond issuers in the portfolio.

3. It is challenging but meaningful to develop dynamic copulas for modeling
high dimensional data in further research.
Bibliography


List of Abbreviations

AIC  Akaike Information Criterion  
ARCH  AutoRegressive Conditional Heteroskedasticity  
BIC  Bayesian Information Criterion  
CDF  Cumulative Distribution Function  
CDO  Collateralized Debt Obligation  
CML  Canonical Maximum Likelihood  
EGARCH  Exponential GARCH  
EURIBOR  Euro Inter-Bank Offered Rate  
FSE  Frankfurt Stock Exchange  
IFM  Inference Functions for Margins  
i.i.d.  Independent and identically distributed  
IGARCH  Integrated GARCH  
GARCH  Generalized ARCH  
KS test  Kolmogorov–Smirnov test  
LIBOR  London Inter-Bank Offered Rate  
LHP  Large Homogeneous Portfolio approximation  
MLE  Maximum Likelihood Estimator  
MRS  Markov Regime Switching  
NIG  Normal Inverse Gaussian  
PDF  Probability Density Function  
RFL  Random Factor Loading  
SJC  Symmetrized Joe-Clayton  
VaR  Value at Risk
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